Abelian groups are to abelian categories as Hilbert spaces are to what?

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The category of Hilbert spaces and bounded linear maps

Hilbert spaces

An *inner product* on a vector space encodes geometry.

$$\|x\| = \sqrt{\langle x|x\rangle} \qquad\qquad \cos\theta = \frac{\langle x|y\rangle}{\|x\|\|y\|}$$

A *Hilbert space* is a vector space with a *complete* inner product.

Every n-dimensional (complex) Hilbert space is isomorphic to $\mathbb{C}^n$ with

$$\big\langle (x_1, x_2, \dots, x_n) \big| (y_1, y_2, \dots, y_n) \big\rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$

$\ell_2(\mathbb{N}) = \big\{ (x_1, x_2, \dots) \in \mathbb{C}^{\mathbb{N}} \;\big|\; |x_1|^2 + |x_2|^2 + \cdots < \infty \big\}$ with

$$\big\langle (x_1, x_2, \dots) \big| (y_1, y_2, \dots) \big\rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \dots$$

Bounded linear maps

A map $T\colon X \to Y$ is ***bounded*** if there is a $C > 0$ such that

$$\|Tx\| \leqslant C\|x\|.$$

A linear map is ***continuous*** if and only if it is bounded.

$\mathbf{Hilb}_{\mathbb{K}}$ is the category of Hilbert spaces and bounded linear maps over $\mathbb{K}$ where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Adjoints

The ***adjoint*** of a bounded linear map $T\colon X \to Y$ is the unique bounded linear map $T^\dagger\colon Y \to X$ such that

$$\langle y|Tx\rangle = \langle T^\dagger y|x\rangle.$$

The matrix of $T^\dagger\colon \mathbb{C}^m \to \mathbb{C}^n$ is the conjugate-transpose of the matrix of $T\colon \mathbb{C}^n \to \mathbb{C}^m$.

Dagger categories

A ***dagger category*** is a category equipped with a choice of $f^\dagger : Y \to X$ for each $f : X \to Y$, such that

$$1^\dagger = 1, \qquad (gf)^\dagger = f^\dagger g^\dagger, \qquad (f^\dagger)^\dagger = f.$$

Examples include $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ and $\mathbf{Hilb}_{\mathbb{H}}$.

Characterisation of $\mathbf{Hilb}_{\mathbb{K}}$

Theorem (Heunen–Kornell[1], Tobin[2])

A dagger category is equivalent to $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ *or* $\mathbf{Hilb}_{\mathbb{H}}$ *if and only if*

- *it has a zero object,*
- *it has binary dagger products,*
- *it has dagger equalisers,*
- *every dagger mono is normal,*
- *the wide subcategory of dagger monos has directed colimits, and*
- *it has a simple separator.*

[1] Heunen and Kornell, "Axioms for the category of Hilbert spaces".
[2] Tobin, "Characterisations for the category of Hilbert spaces".

Dagger monos

A linear map $f : X \to Y$ is an ***isometry*** if $\|fx\| = \|x\|$.

Isometries represent ***closed*** subspaces.

A morphism $f : X \to Y$ is ***dagger monic*** if $f^\dagger f = 1$.

A bounded linear map is an isometry if and only if it is dagger monic.

Dagger kernels and equalisers

The ***kernel*** of a bounded linear map $f\colon X \to Y$ is the subspace

$$\mathrm{Ker}\, f = \{x \in X \mid fx = 0\}.$$

The restricted inner product makes $\mathrm{Ker}\, f$ a Hilbert space and the canonical inclusion $\mathrm{Ker}\, f \hookrightarrow X$ an isometry.

A ***dagger kernel***/***equaliser*** is a dagger monic kernel/equaliser.

Hilb has dagger kernels/equalisers.

Dagger coproducts

The ***direct sum*** of X and Y is

$$X \oplus Y = \{(x,y) \mid x \in X, y \in Y\},$$
$$\langle (x,y) | (x',y') \rangle = \langle x|x' \rangle + \langle y|y' \rangle.$$

The injections

$$i_1 \colon X \to X \oplus Y \qquad X \oplus Y \leftarrow Y : i_2$$
$$x \mapsto (x,0) \qquad (0,y) \leftarrow\!\shortmid y$$

are ***orthogonal*** isometries.

A ***dagger coproduct*** is a coproduct whose injections are dagger monic and pairwise orthogonal.

$$i_j^\dagger i_k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Dagger coproducts are biproducts.

Hilb has finite dagger coproducts.

Characterisations of $\mathbf{Hilb}_{\mathbb{K}}$ and $\mathbf{Mod}_R$

Theorem (Heunen–Kornell, Tobin)

A dagger category is equivalent to $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ or $\mathbf{Hilb}_{\mathbb{H}}$ if and only if

- *it has a zero object,*
- *it has binary dagger products,*
- *it has dagger kernels,*
- *binary diagonals are normal,*
- *the subcategory of dagger monos has directed colimits,*
- *it has a simple separator.*

Theorem

A category is equivalent to $\mathbf{Mod}_R$ for some ring R if and only if

- *it has a zero object,*
- *it has binary products/coproducts,*
- *it has kernels/cokernels,*
- *all monos/epis are normal,*
- *it has small coproducts,*
- *it has a compact projective separator.*[3]

[3] See Freyd, “Abelian Categories”, p. 106.

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Rational dagger categories

Definition

A dagger category is ***rational*** if

- it has a zero object,
- it has binary dagger products,
- it has dagger kernels, and
- all $\Delta : X \to X \oplus X$ are normal.

A category is ***abelian*** if

- it has a zero object,
- it has binary products/coproducts,
- it has kernels/cokernels, and
- all monos/epis are normal.[4]

[4] See Borceux, *Handbook of Categorical Algebra*.

Examples of rational dagger categories

- **Hilb**$_{\mathbb{K}}$ where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.
- For each W*-algebra A, the category of self-dual Hilbert A-modules and bounded A-linear maps.
- For each partially semiordered involutive division ring R, the category of finite-dimensional inner-product R-modules and R-linear maps.
- For each involutive division ring R that is formally complex and quadratically closed, the category of R-valued matrices.

Similar properties

Rational dagger category properties:

- monic if and only if zero kernel,
- (epi, dagger mono) factorisations,
- additive, so have finite dagger biproducts and dagger equalisers,
- normal monos are pushout stable,
- pushouts along normal monos are pullbacks.

Abelian category properties:

- monic if and only if zero kernel,
- (epi, mono) factorisations,
- additive, so have finite biproducts and equalisers,
- monos are pushout stable,
- pushouts along monos are pullbacks.

Additivity

Theorem

In a semiadditive category, an object X is abelian if and only if $\nabla : X \oplus X \to X$ is the cokernel of a split mono.

Let $e\colon X \oplus X \to X \oplus X$ be the induced idempotent. Then

$$(p_1 + p_2)e = 0 \qquad \text{and} \qquad fe = 0 \implies fi_1 = fi_2.$$

Observe that

$$\begin{aligned}(1 + p_1ei_2 + p_2ei_1)p_1e &= p_1e + p_1ei_2p_1e + p_2ei_1p_1e + p_2ei_2(p_1 + p_2)e \\ &= p_1e + (p_1 + p_2)ei_2p_1e + p_2e(i_1p_1 + i_2p_2)e = p_1e + p_2e^2 = 0.\end{aligned}$$

Hence $1 + p_1ei_2 + p_2ei_1 = (1 + p_1ei_2 + p_2ei_1)p_1i_1 = (1 + p_1ei_2 + p_2ei_1)p_1i_2 = 0.$

Partial semiordering

Let **C** be a rational dagger category.

For each object A of **C**, define $\leqslant$ on the self-adjoint endomorphisms of A by

$$a \leqslant b \qquad \Longleftrightarrow \qquad b - a = x^\dagger x \text{ for some } x\colon A \to X.$$

Then $\leqslant$ is a partial order with the following properties:

$$0 \leqslant 1, \qquad a \leqslant b \implies a + c \leqslant b + c, \qquad a \leqslant b \implies f^\dagger a f \leqslant f^\dagger b f.$$

Each endohomset of **C** is thus a *partially semiordered involutive ring*.

Theorem

In a rational dagger category, if $a \geqslant 1$ then a is invertible.[5]

Theorem

Rational dagger categories are uniquely enriched in the category of rational vector spaces.

[5] Similar to Handelman, "Rings with involution as partially ordered abelian groups", Proposition 1.13.

Orthogonal complements

The ***orthogonal complement*** of a mono $m \colon A \to X$ is the dagger mono $m^\perp \colon X \ominus A \to X$ defined by

$$m^\perp = \ker m^\dagger = (\operatorname{coker} m)^\dagger.$$

Well-known properties of kernels and cokernels imply that

$$m^{\perp\perp\perp} = m^\perp, \qquad 0^\perp = 1, \qquad 1^\perp = 0, \qquad m \leqslant n^\perp \iff m^\perp \leqslant n.$$

Theorem

In a rational dagger category, if $m \colon A \to X$ is dagger monic, then $(X, m, m^\perp)$ is a dagger coproduct of A and $X \ominus A$.

Gram–Schmidt procedure

In a rational dagger category, for each biproduct

$$\left(A_k \underset{r_k}{\overset{s_k}{\rightleftarrows}} X\right)_{k=1}^{n},$$

the equations

$$t_1 = s_1 \qquad \text{and} \qquad t_{m+1} = s_{m+1} - \sum_{k=1}^{m} t_k ({t_k}^{\dagger} t_k)^{-1} {t_k}^{\dagger} s_{m+1}.$$

define an ***orthogonal*** biproduct

$$\left(A_k \underset{({t_k}^{\dagger} t_k)^{-1} {t_k}^{\dagger}}{\overset{t_k}{\rightleftarrows}} X\right)_{k=1}^{n}$$

where $\bigcup_{k=1}^{m} t_k = \bigcup_{k=1}^{m} s_k$ for each m.

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Conclusion

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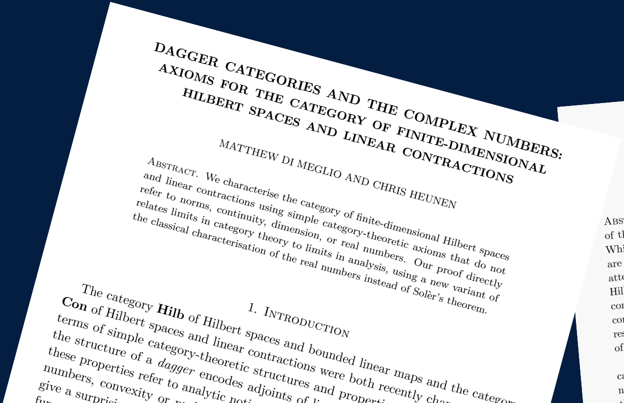


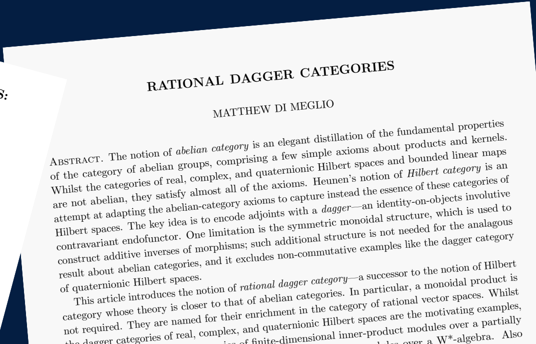
DAGGER CATEGORIES AND THE COMPLEX NUMBERS: AXIOMS FOR THE CATEGORY OF FINITE-DIMENSIONAL HILBERT SPACES AND LINEAR CONTRACTIONS

MATTHEW DI MEGLIO AND CHRIS HEUNEN

Abstract. We characterise the category of finite-dimensional Hilbert spaces and linear contractions using simple category-theoretic axioms that do not refer to norms, continuity, dimension, or real numbers. Our proof directly relates limits in category theory to limits in analysis, using a new variant of the classical characterisation of the real numbers instead of Solèr's theorem.

1. Introduction

The category **Hilb** of Hilbert spaces and bounded linear maps and the category
Con of Hilbert spaces and linear contractions were both recently char
terms of simple category-theoretic structures and properti
the structure of a *dagger* encodes adjoints of l
these properties refer to analytic noti
numbers, convexity or
give a surprisi



RATIONAL DAGGER CATEGORIES

MATTHEW DI MEGLIO

Abstract. The notion of *abelian category* is an elegant distillation of the fundamental properties of the category of abelian groups, comprising a few simple axioms about products and kernels. Whilst the categories of real, complex, and quaternionic Hilbert spaces and bounded linear maps are not abelian, they satisfy almost all of the axioms. Heunen's notion of *Hilbert category* is an attempt at adapting the abelian-category axioms to capture instead the essence of these categories of Hilbert spaces. The key idea is to encode adjoints with a *dagger*—an identity-on-objects involutive contravariant endofunctor. One limitation is the symmetric monoidal structure, which is used to construct additive inverses of morphisms; such additional structure is not needed for the analogous result about abelian categories, and it excludes non-commutative examples like the dagger category of quaternionic Hilbert spaces.

This article introduces the notion of *rational dagger category*—a successor to the notion of Hilbert category whose theory is closer to that of abelian categories. In particular, a monoidal product is not required. They are named for their enrichment in the category of rational vector spaces. Whilst the dagger categories of real, complex, and quaternionic Hilbert spaces are the motivating examples,
of finite-dimensional inner-product modules over a partially
over a W*-algebra. Also

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