# Universal indexed categories

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TACL 2022

1. Introduction to indexed categories
2. Universality of self indexing
3. Indexed monoidal categories

# Families of objects and morphisms are *ubiquitous* and *fundamental* in category theory

| Concept | Family |
|---|---|
| category $\underline{C}$ | $\left(\underline{C}(X,Y)\right)_{X,Y\in \mathrm{ob}\,\underline{C}}$ |
| functor $\underline{C} \xrightarrow{F} \underline{D}$ | $\left(\underline{C}(X,Y) \xrightarrow{F_{X,Y}} \underline{D}(FX,FY)\right)_{X,Y\in \mathrm{ob}\,\underline{C}}$ |
| natural transformation $\begin{tikzcd} \underline{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \underline{D} \arrow[from=U, to=D, Rightarrow, "\tau"] \end{tikzcd}$ | $\left(FX \xrightarrow{\tau_X} GX\right)_{X\in \mathrm{ob}\,\underline{C}}$ |

# Indexed categories

$\underline{S}$ = category with a terminal object $1$

An $\underline{S}$-indexing $\mathbb{C}$ of a category $\underline{C}$ consists of

- for all $J$ in $\underline{S}$, a category $\mathbb{C}^J$ with $\mathbb{C}^1 \cong \underline{C}$
  (objects and morphisms of $\mathbb{C}^J$ are J-indexed families of those of $\underline{C}$)

- for all $r : J \to K$ in $\underline{S}$, a functor $\Delta_r : \mathbb{C}^K \to \mathbb{C}^J$
  ($\Delta_r$ reindexes K-indexed families along $r$)

such that $\Delta_{rs} \cong \Delta_s \Delta_r$ and $\Delta_{id_J} \cong id_{\mathbb{C}^J}$ coherently.
(i.e. the data forms a pseudofunctor $\underline{S}^{op} \longrightarrow \underline{Cat}$)

<u>Example</u> (set indexing)

$\underline{S} = \underline{Set}$

$\mathbb{C}^J = \prod_{j \in J} \underline{C}$

$\Delta_r (Y_k)_{k \in K} = (Y_{rj})_{j \in J}$

<u>Example</u> (self indexing)

$\underline{S}$ = finitely complete $\underline{C}$

$\mathbb{C}^J = \underline{C}/J$

$\Delta_r (Y \xrightarrow{y} J)$ = chosen pullback of $y$ along $r$

# Indexed sums

An indexed category has **indexed sums** if each $\Delta_r$ has a left adjoint $\Sigma_r$ that is compatible with reindexing.

## Proposition

A category has small sums if and only if its set indexing has indexed sums.

$$\Sigma_r (X_j)_{j\in J} = \Big(\sum_{j\in r^{-1}\{k\}} X_j\Big)_{k\in K}$$

## Proposition

The self indexing of a finitely complete category has indexed sums.

$$\Sigma_r (X \xrightarrow{x} J) = (X \xrightarrow{rx} K)$$

# Extensivity

An indexed category $\underline{S}^{op} \xrightarrow{\mathbb{C}} \underline{Cat}$ is extensive if

- it has indexed sums, and
- for all $r: J \longrightarrow K$ in $\underline{S}$ and all $X$ in $\mathbb{C}^J$, the functor

$$\Sigma_r : \mathbb{C}^J/X \longrightarrow \mathbb{C}^K/\Sigma_r X$$

is an equivalence of categories.

## Proposition

A category is extensive if and only if its set indexing is too.

## Proposition

The self indexing of a finitely complete category is extensive.

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# Universality of self indexing



$$\left(\underline{S}^{op} \xrightarrow{\mathbb{C}} \underline{Lex}\right) \longmapsto \mathbb{C}^1$$

$$\begin{tikzcd} \underline{ExtIndLex} \arrow[r, bend left, "\text{underlying category}"] \arrow[r, phantom, "\perp"] & \underline{Lex} \arrow[l, bend left, "\text{self indexing}"] \end{tikzcd}$$

$$\left(\underline{D}^{op} \xrightarrow{\underline{D}/(-)} \underline{Lex}\right) \longleftarrow\!\!| \ \underline{D}$$

Extensive finitely complete indexed categories and finitely continuous indexed functors that preserve indexed coproducts

# Universality of self indexing



$$
\begin{tikzcd}
\underline{\mathrm{ExtIndLex}} \arrow[rr, bend left, "\text{underlying category}"] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{Lex}} \arrow[ll, bend left, "\text{self indexing}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\underline{S}^{op} \arrow[r, "\mathbb{C}"] & \underline{\mathrm{Lex}}\right) \arrow[d, dashed, "\exists!\,(F,\,G^{(-)})"'] \\
\left(\underline{D}^{op} \arrow[r, "\underline{D}/(-)"'] & \underline{\mathrm{Lex}}\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}^1 \arrow[d, dashed, "G^1"'] \arrow[dr, "\forall H"] & \\
\underline{D}/1 \arrow[r, "\simeq", "\varepsilon_{\underline{D}}"'] & \underline{D}
\end{tikzcd}
$$

## Existence

$$
\begin{tikzcd}
\underline{S} \arrow[d, "F"'] & J \arrow[rr, "r"] \arrow[d, mapsto] & & K \arrow[d, mapsto] \\
\underline{D} & H\Sigma_J 1_J \cong H\Sigma_K \Sigma_r \Delta_r 1_K \arrow[rr, "H\Sigma_K \varepsilon_r 1_K"'] & & H\Sigma_K 1_K
\end{tikzcd}
$$

$$
G^J = \left(\mathbb{C}^J \cong \mathbb{C}^J/1_J \xrightarrow{\Sigma_J} \mathbb{C}^1/\Sigma_J 1_J \xrightarrow{H} \underline{D}/FJ\right)
$$

## Remarks

- Easier to prove strict functoriality and uniqueness using fibrations
- Need extensivity to show that $G$ is compatible with $\Delta$
- Construction same as Moens (1982)

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# Indexed monoidal categories



A symmetric monoidal $\underline{S}$-indexing of a symmetric monoidal category $\mathcal{V}$ is a pseudofunctor

$$\mathbb{V} : \underline{S}^{op} \longrightarrow \underline{\mathrm{SymMonCat}}$$

strong symmetric monoidal functors

where $\underline{S}$ is cartesian monoidal and $\mathbb{V}^1 \overset{n}{\cong} \mathcal{V}$.

## Example (set indexing)

$\underline{S} = \underline{\mathrm{Set}}$ $\qquad \mathbb{V}^J = \prod_{j \in J} \mathcal{V}$ $\qquad (X_j)_{j\in J} \otimes_J (Y_j)_{j \in J} = (X_j \otimes Y_j)_{j \in J}$

## Non-example (self indexing)

$\underline{S} = \mathcal{V}$ $\qquad \mathbb{V}^J = \mathcal{V}/J$ $\qquad (X \xrightarrow{x} J) \otimes_J (Y \xrightarrow{y} J) = ?$

# Comonoids

A (cocommutative) comonoid $J$ is an object $J$ equipped with morphisms

$$d_J : J \longrightarrow J \otimes J \qquad \text{and} \qquad e_J : J \longrightarrow I$$

comultiplication          counit

subject to coassociativity, counitality and cocommutativity laws.

$\underline{\mathrm{Comon}}_{\mathcal{V}}$ = category of comonoids in $\mathcal{V}$

**Proposition**

If $\mathcal{V}$ is cartesian monoidal, then $\underline{\mathrm{Comon}}_{\mathcal{V}} \cong \mathcal{V}$

**Proposition** (Fox 1975)

$\underline{\mathrm{Comon}}_{\mathcal{V}}$ is the cofree cartesian monoidal category on $\mathcal{V}$

# Comodules



A J-comodule $(X, x)$ is an object $X$ equipped with a morphism

$$\underbrace{x : X \longrightarrow J \otimes X}_{\text{coaction}}$$

that preserves comultiplication and counits.

$$\underline{\mathrm{Comod}}_{\mathcal{V}}(J) = \text{category of } J\text{-comodules in } \mathcal{V}$$

## Proposition

If $\mathcal{V}$ is cartesian monoidal, then $\underline{\mathrm{Comod}}_{\mathcal{V}}(J) \cong \mathcal{V}/J$

# Comonoid indexing

The comonoid indexing of a symmetric monoidal category $\mathcal{V}$ with nice equalisers is the indexed category

$$\underline{\mathrm{Comod}}_{\mathcal{V}}(-) : \underline{\mathrm{Comon}}_{\mathcal{V}}^{op} \longrightarrow \underline{\mathrm{SymMonCat}}$$

## Proposition

If $\mathcal{V}$ is cartesian monoidal, then it is finitely complete and its comonoid indexing is isomorphic with its self indexing.

# Monoidal extensivity

A symmetric monoidal category $\mathcal{V}$ is (infinitary) monoidal extensive if it has small coproducts and the functor

$$\sum_{j\in J} : \prod_{j\in J} \underline{\mathrm{Comod}}_{\mathcal{V}}(A_j) \longrightarrow \underline{\mathrm{Comod}}_{\mathcal{V}}\Big(\sum_{j\in J} A_j\Big)$$

is always an equivalence of categories.

Example (Grunenfelder-Paré 1987) $\underline{\mathrm{Vect}}_{\mathbb{k}}$

An indexed symmetric monoidal category $\mathbb{V} : \underline{S}^{op} \longrightarrow \underline{\mathrm{SymMonCat}}$ is monoidal extensive if it has indexed coproducts and the functor

$$\Sigma_r : \mathrm{Comod}_{\mathbb{V}J}(A) \longrightarrow \mathrm{Comod}_{\mathbb{V}K}(\Sigma_r A)$$

is always an equivalence of categories.

# Universality of comonoid indexing *

$$
\begin{tikzcd}
\underline{\mathrm{ExtEqIndSymMonCat}} \arrow[rr, bend left, "\text{underlying category}"] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{EqSymMonCat}} \arrow[ll, bend left, "\text{comonoid indexing}"]
\end{tikzcd}
$$

* work in progress

## Conclusion

- Comonoid indexing of nice symmetric monoidal categories generalises self indexing of finitely complete categories

- Taking the self indexing/comonoid indexing is right adjoint to taking the underlying category

- There is a monoidal generalisation of extensivity

## Future work

- Finish checking details for monoidal case

- Check monoidal version of Moens' theorem

- Investigate links to linear dependent types

- Work out link between categories internal to and enriched in a monoidal category via monoidal extensivity