Categorical Hilbert theory

Matthew Di Meglio

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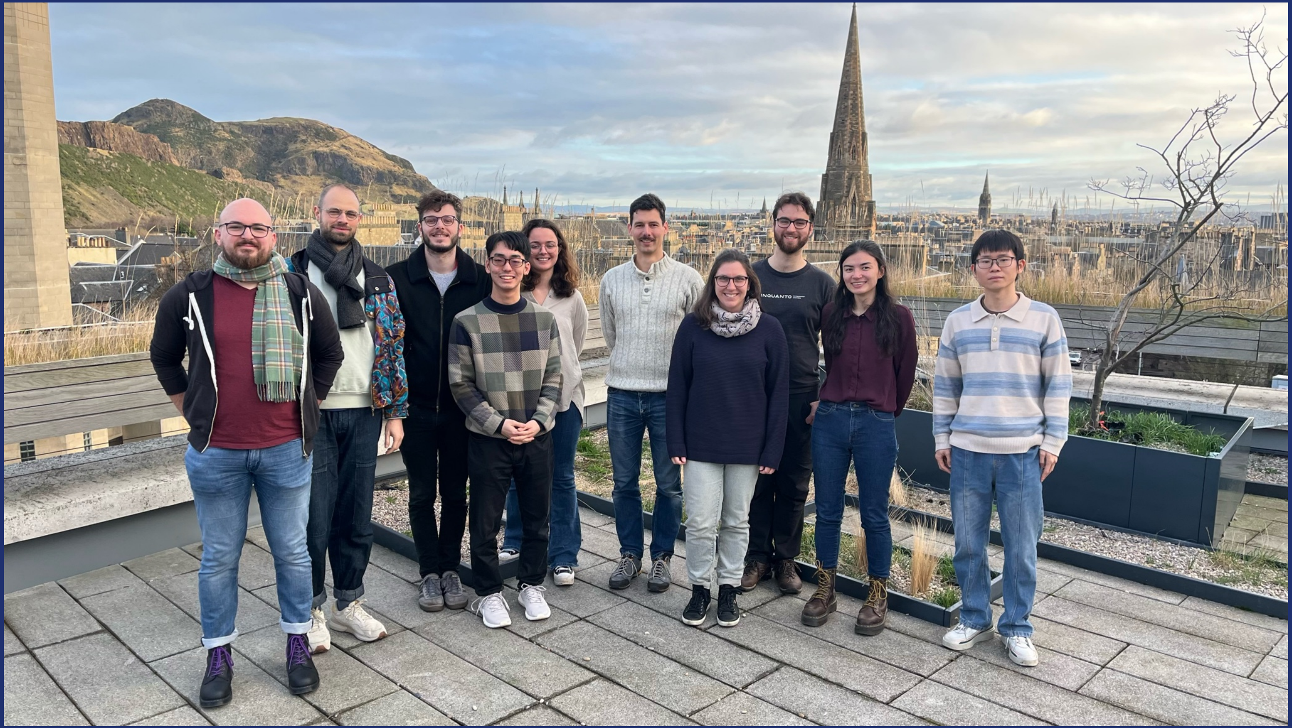
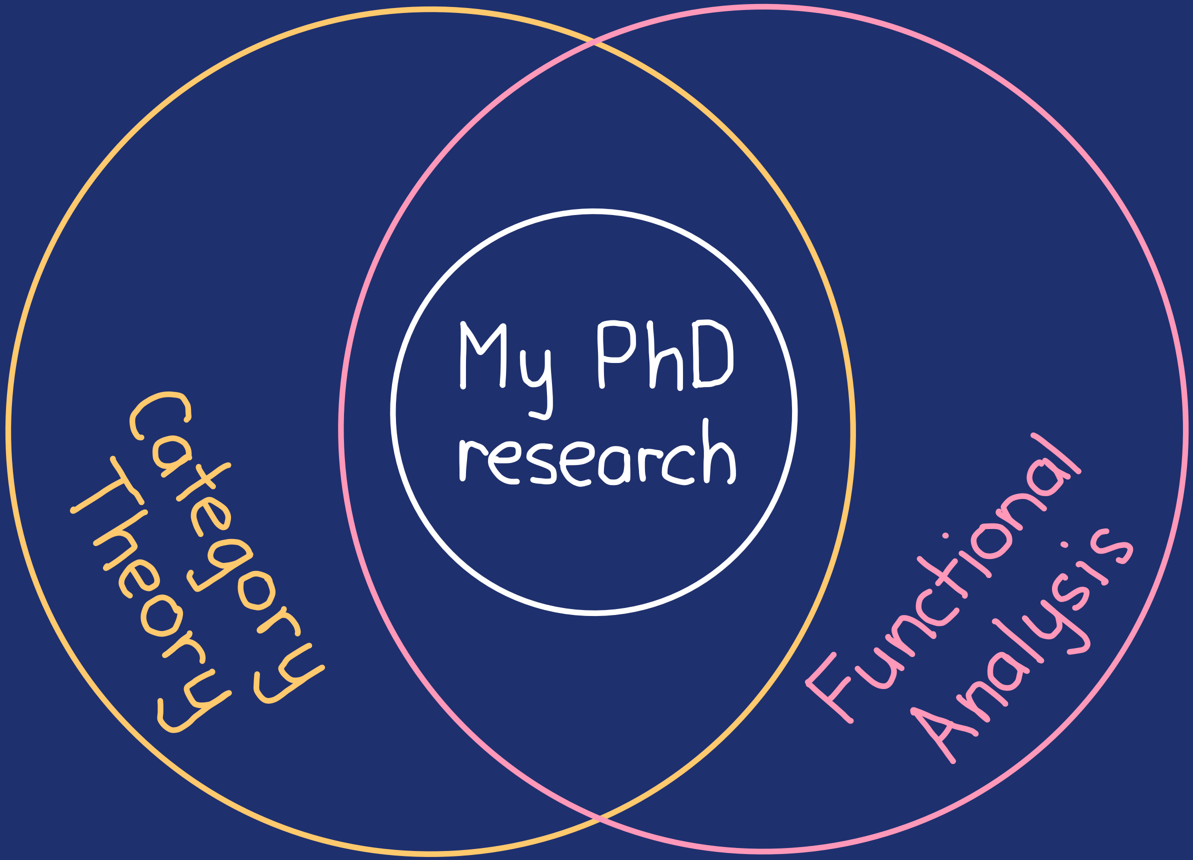
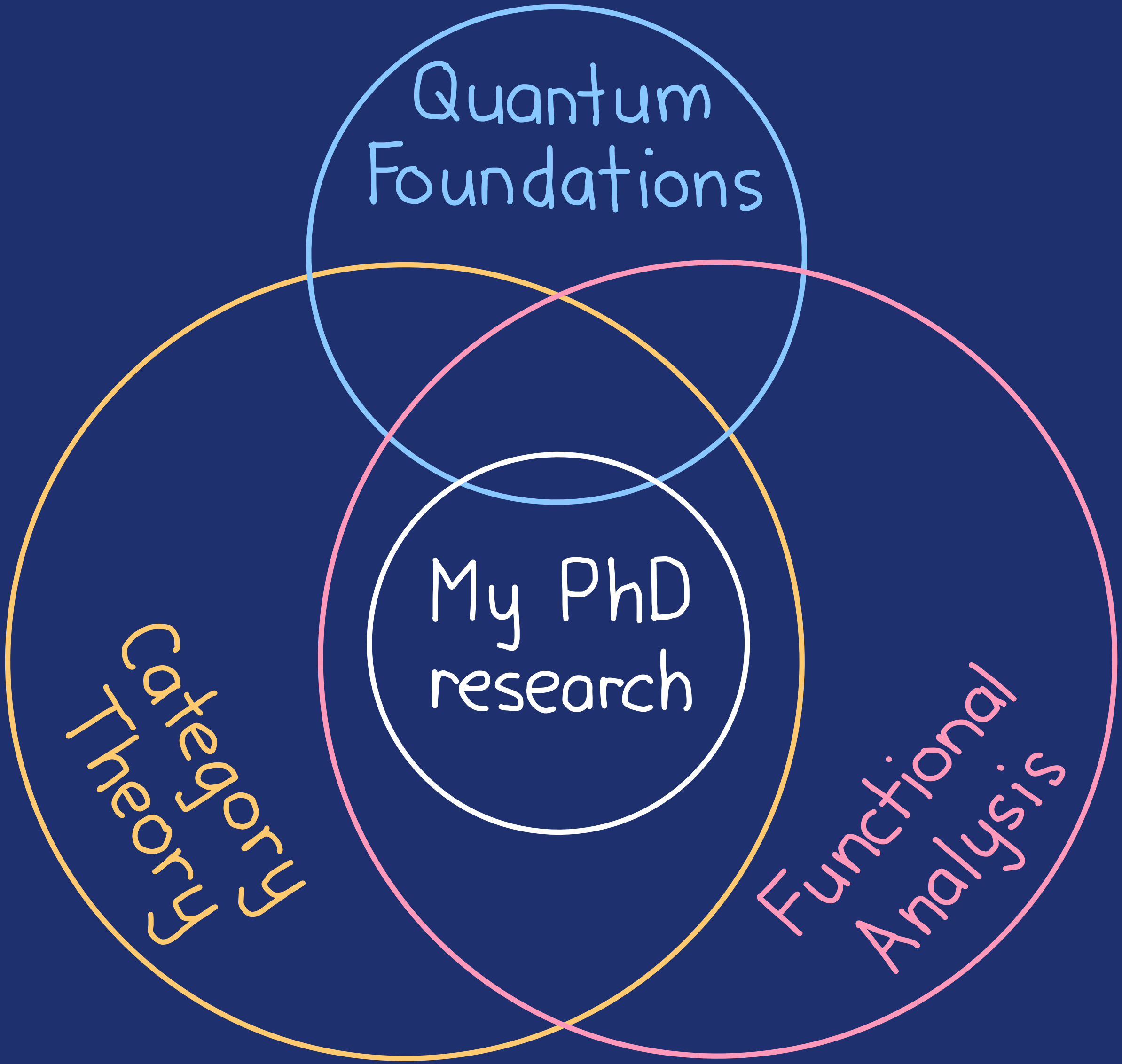


PhD student of Chris Heunen at the University of Edinburgh

My PhD research


My PhD research
Category Theory

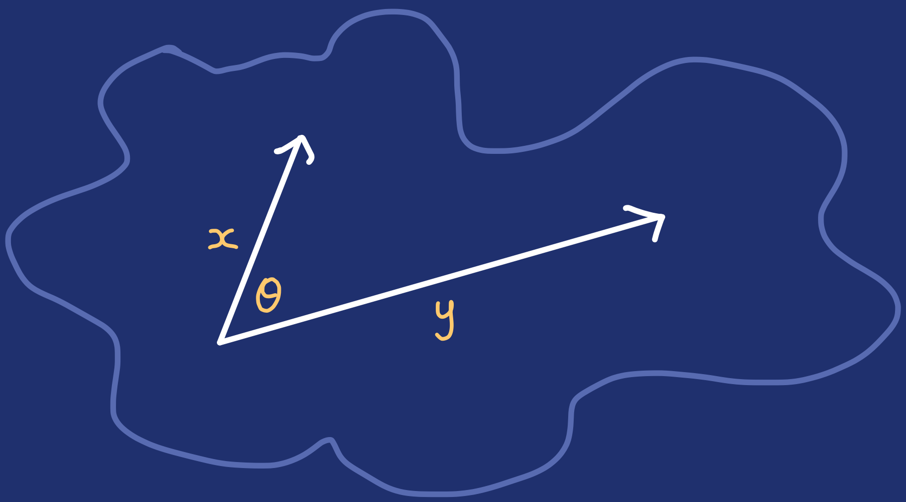

My PhD research
Category Theory
Functional Analysis


Quantum Foundations
My PhD research
Category Theory
Functional Analysis

1. Limits are limits
2. Categorical Hilbert theory
3. Dilators

Background

Hilbert spaces are vector spaces with geometry (encoded by a complete innerproduct)



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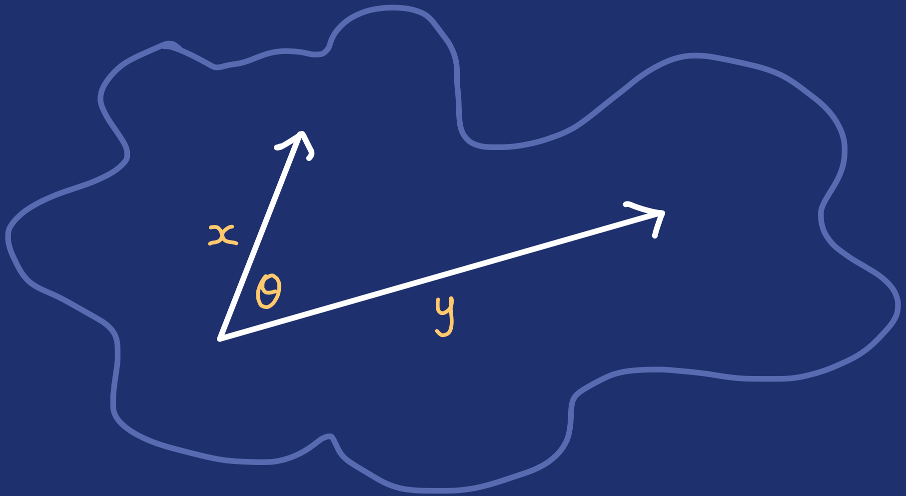
$\|x\| = \sqrt{\langle x | x \rangle}$

Lengths



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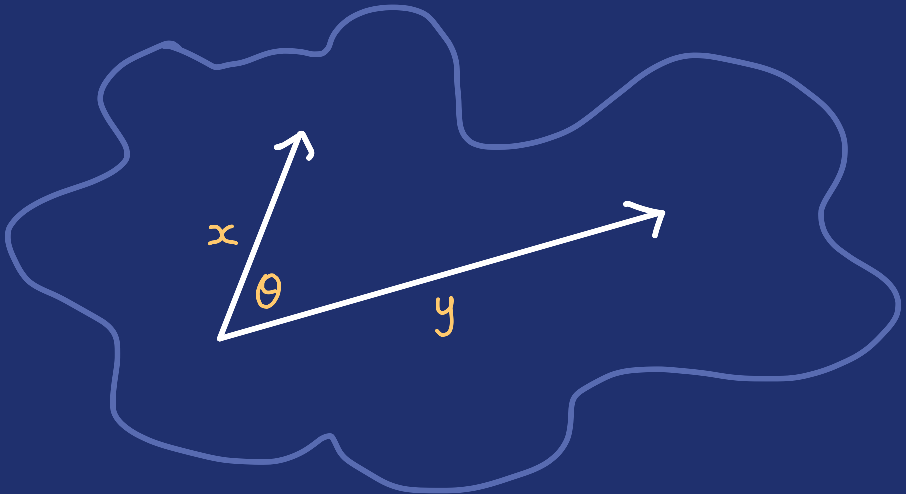

$$\cos\theta = \frac{\operatorname{Re}\langle x | y \rangle}{\|x\| \, \|y\|}$$

Angles

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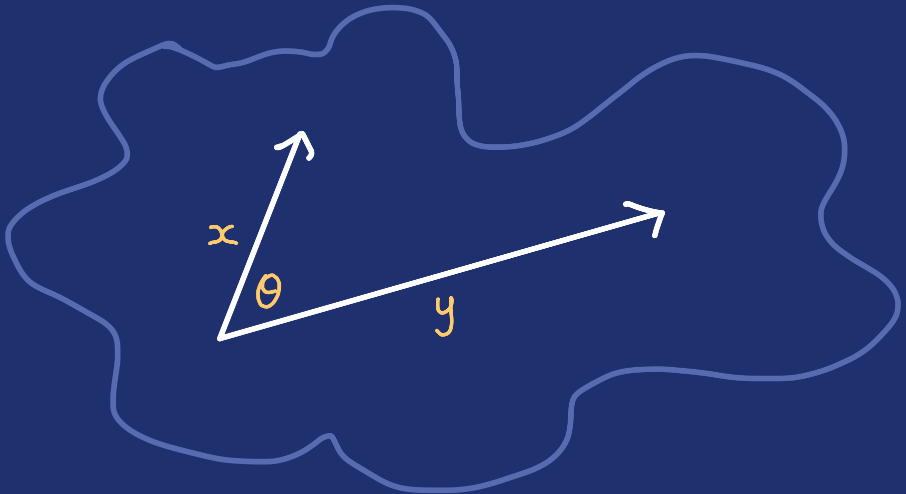

$$\cos\theta = \frac{\mathrm{Re}\,\langle x | y \rangle}{\|x\|\,\|y\|}$$

Angles

e.g. $\mathbb{C}, \mathbb{C}^2, \ldots, \ell^2(\mathbb{N})$

Adjointables are maps $f : X \to Y$ between Hilbert spaces with an adjoint $f^* : Y \to X$

$$\langle y | f x \rangle = \langle f^* y | x \rangle$$

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They form a category **Adjointable**

Contractions are linear maps between Hilbert spaces that decrease lengths

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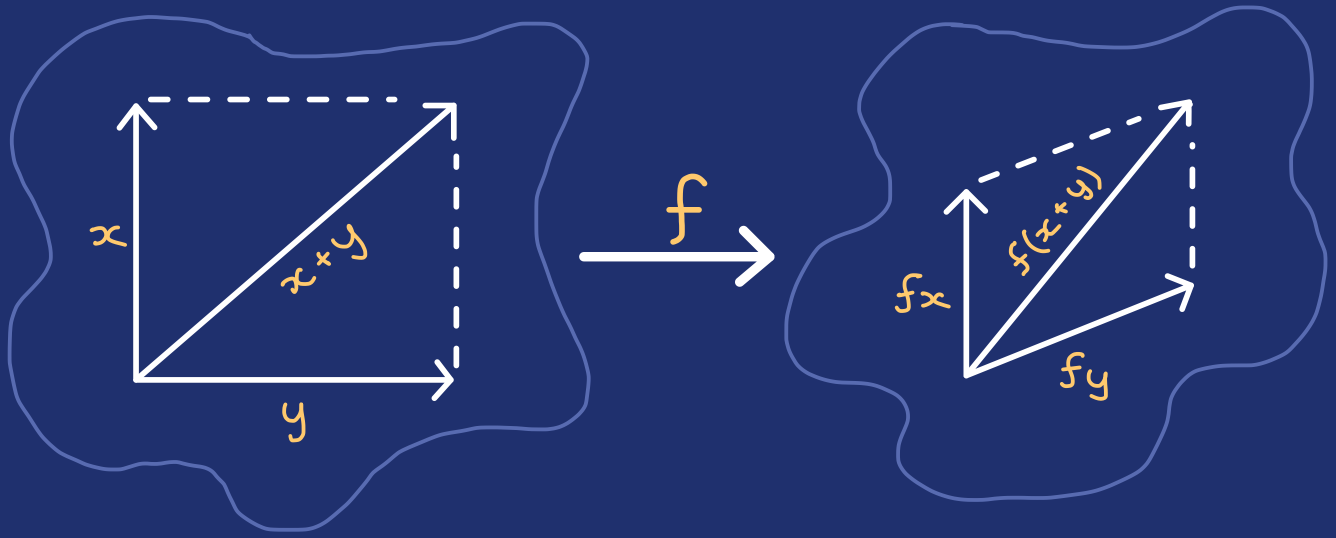

$$\|fx\| \leq \|x\|$$

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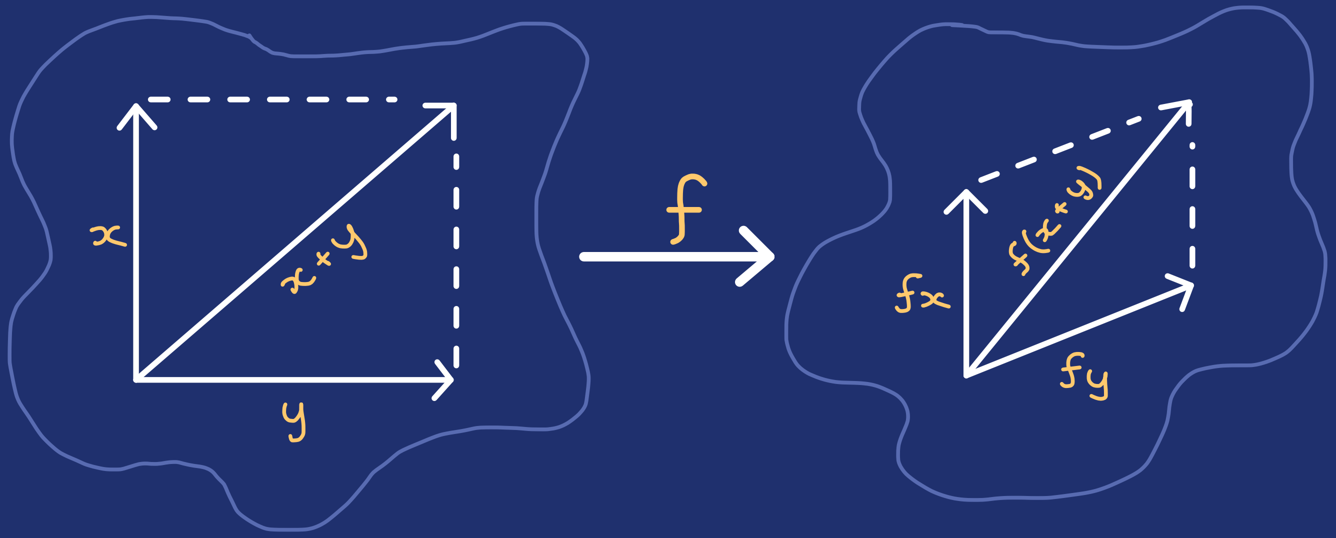

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They form a subcategory **Contraction** of **Adjointable**

Isometries are maps between Hilbert spaces that preserve geometry

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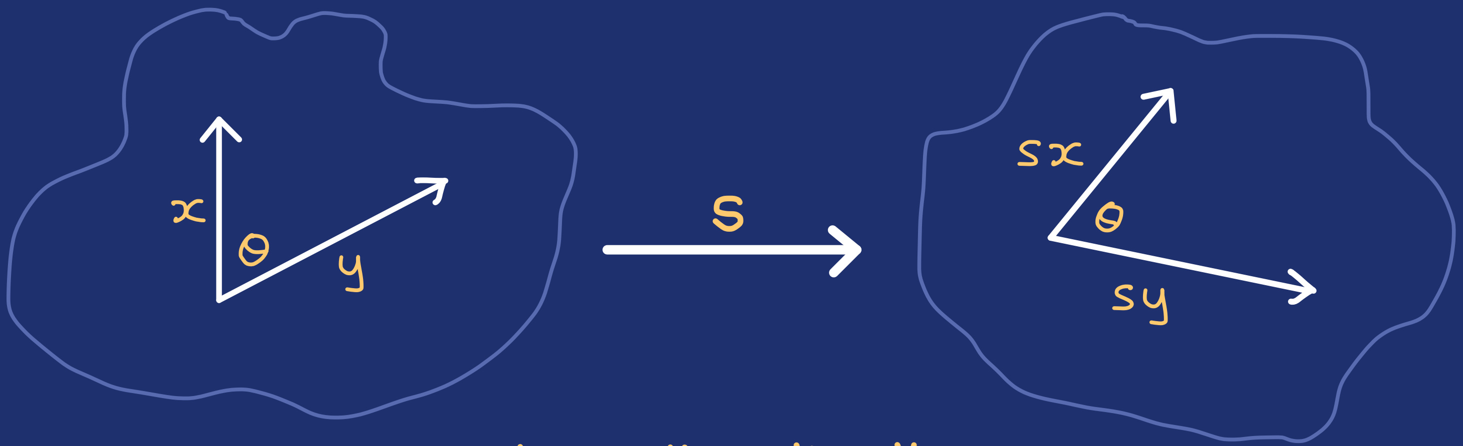

$$\|Sx\| = \|x\|$$

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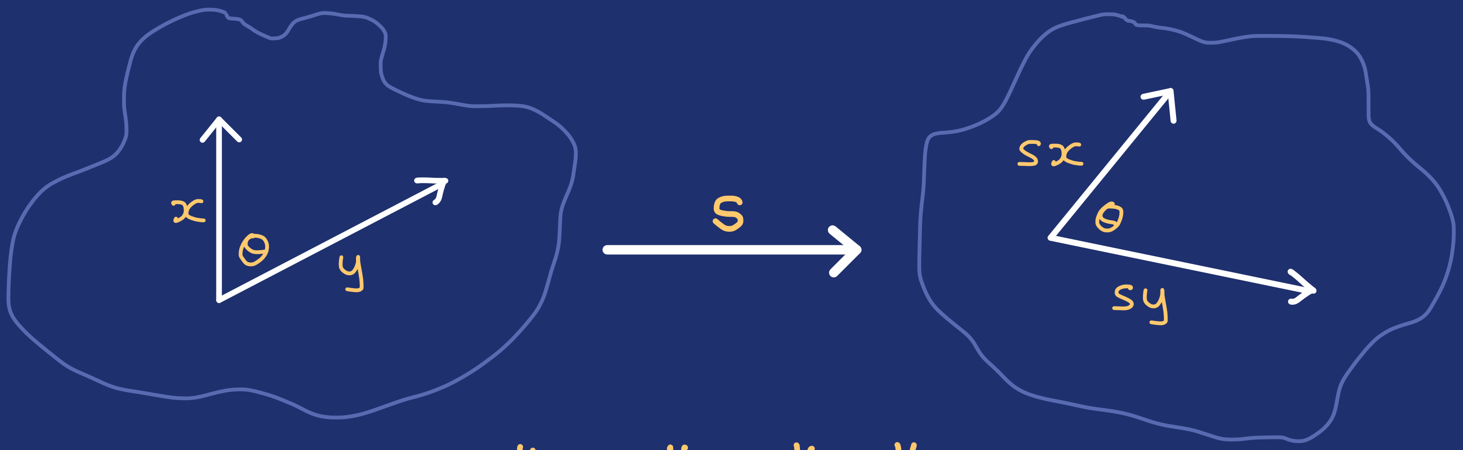

$$\|sx\| = \|x\|$$

They form a subcategory **Isometry** of **Contraction**

① Limits are limits

Limits in analysis are about approximating elements

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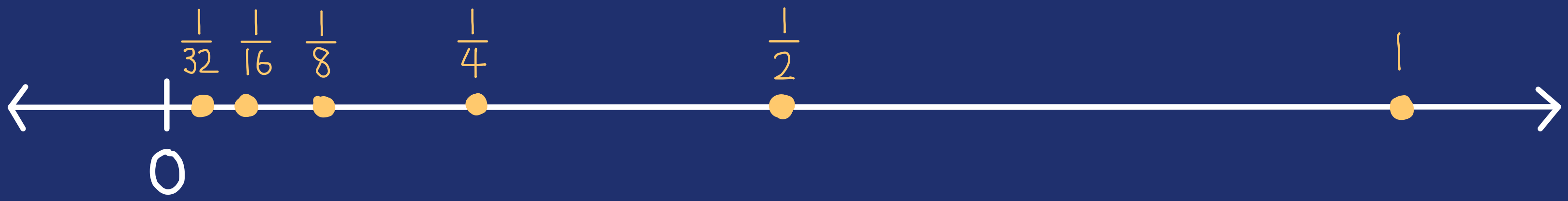

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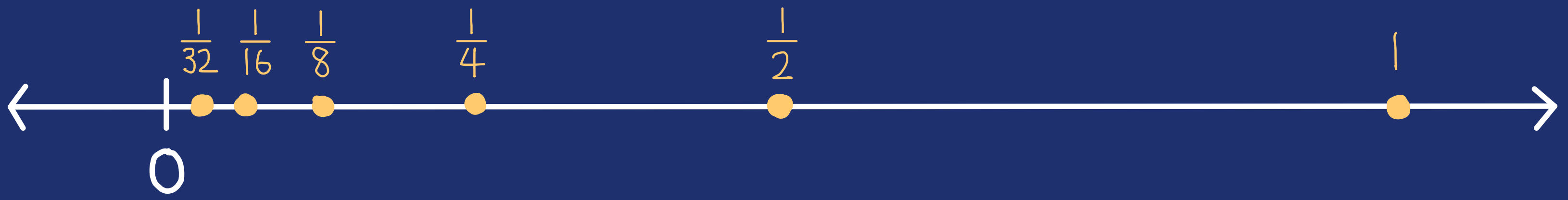

$$\lim_{n\to\infty} 2^{-n} = 0$$

Codirected limits in category theory are about approximating objects

in the category Contraction

Codirected limits in category theory are about approximating objects

$$x_1 \longleftarrow\!\shortmid \quad (x_1, x_2) \quad \longleftarrow\!\shortmid \quad (x_1, x_2, x_3)$$

$$\mathbb{C} \longleftarrow \mathbb{C}^2 \longleftarrow \mathbb{C}^3 \longleftarrow \cdots$$

in the category Contraction

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$$
\begin{array}{ccccccc}
x_1 & \longmapsfrom & (x_1, x_2) & \longmapsfrom & (x_1, x_2, x_3) & & \\
\mathbb{C} & \longleftarrow & \mathbb{C}^2 & \longleftarrow & \mathbb{C}^3 & \longleftarrow & \cdots
\end{array}
$$

$$\lim_{n \in \mathbb{N}} \mathbb{C}^n = \ell^2(\mathbb{N}) = \left\{ x \in \mathbb{C}^{\mathbb{N}} \;\middle|\; |x_1|^2 + |x_2|^2 + \cdots < \infty \right\}$$

in the category Contraction

AXIOMS FOR THE CATEGORY OF HILBERT SPACES

CHRIS HEUNEN AND ANDRE KORNELL

Abstract. We provide axioms that guarantee a category is equivalent to that of continuous linear functions between Hilbert spaces. The axioms are purely categorical and do not presuppose any analytical structure. This addresses a question about the mathematical foundations of quantum theory raised in reconstruction programmes such as those of von Neumann, Mackey, Jauch, Piron, Abramsky, and Coecke.

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$\mathbb{R}$ or $\mathbb{C}$ from codirected limits via black-box: Solèr's theorem

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What is the deeper connection between these two kinds of limits?

KEY IDEA

Given real numbers $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant 1$

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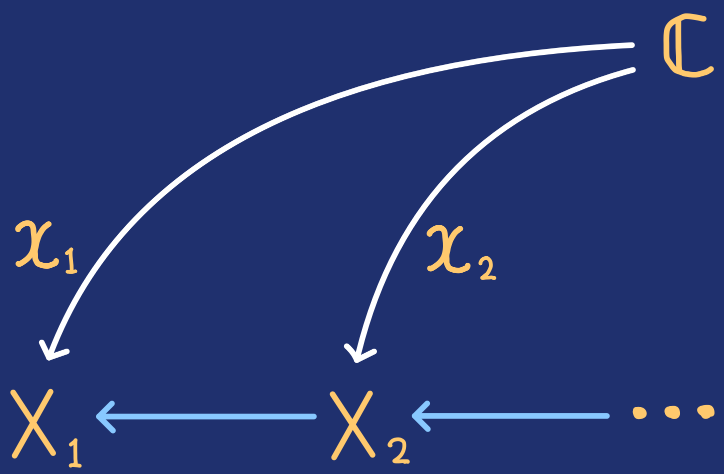

$\|x_1\|^2 = a_1 \qquad \|x_2\|^2 = a_2 \qquad \cdots$

in the category Contraction

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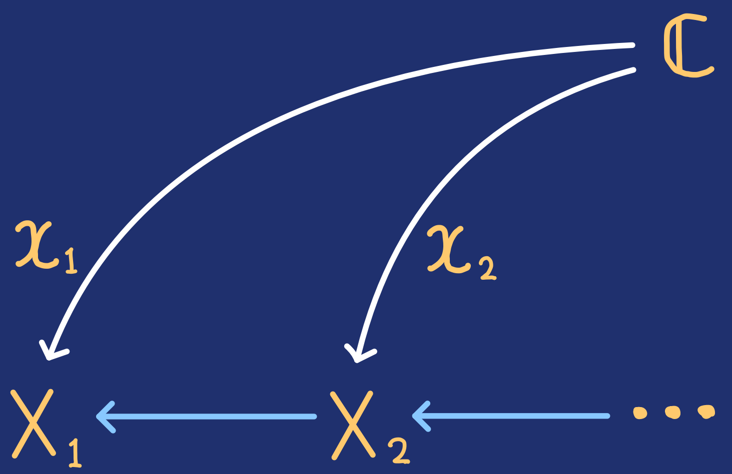
Given real numbers $0 < a_1 \leq a_2 \leq \cdots \leq 1$



$$\|x_1\|^2 = a_1 \qquad \|x_2\|^2 = a_2 \qquad \cdots$$

$$\lim_{n \in \mathbb{N}} X_n$$

in the category Contraction

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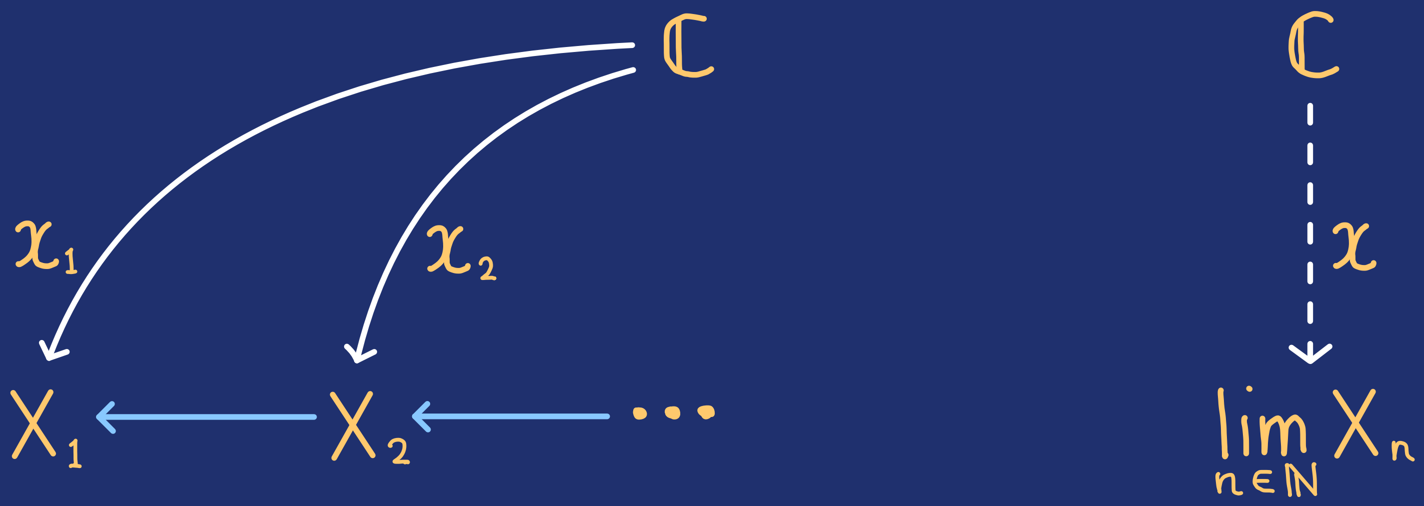

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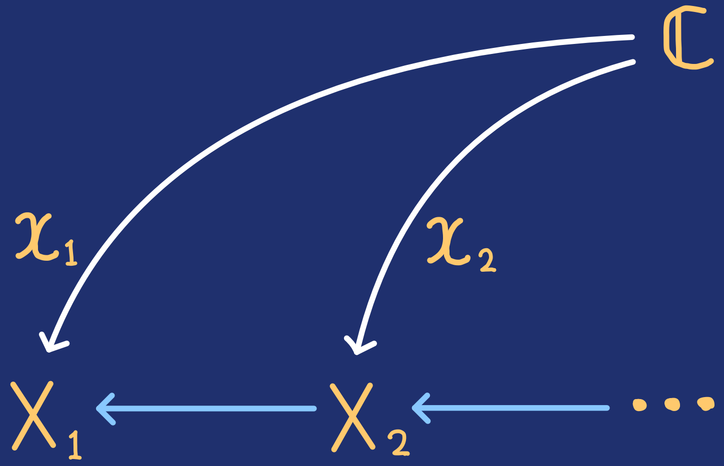
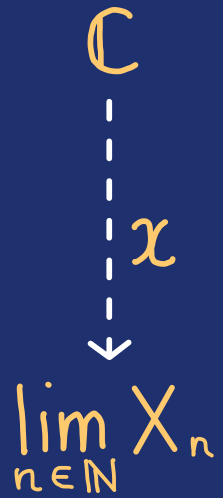


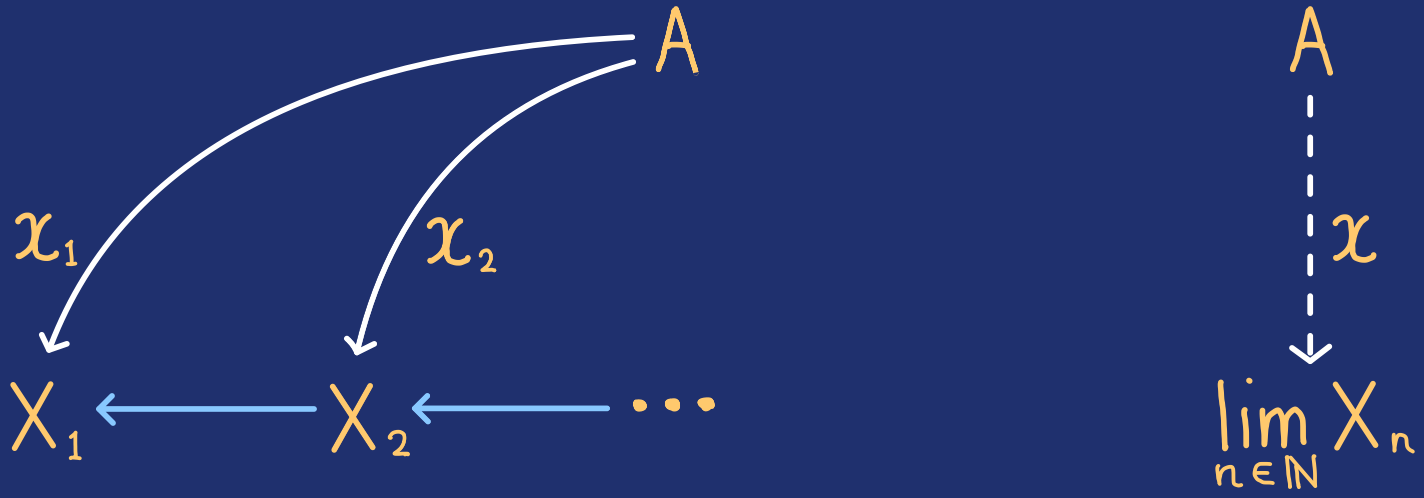


$\|x_1\|^2 = a_1$ $\quad \|x_2\|^2 = a_2$ $\quad \cdots$ $\qquad \|x\|^2 = \sup_{n \in \mathbb{N}} a_n$

in the category **Contraction**

KEY IDEA

Given operators $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant 1$ on a Hilbert space A



$x_1^* x_1 = a_1 \qquad x_2^* x_2 = a_2 \qquad \cdots \qquad x^* x = \sup_{n \in \mathbb{N}} a_n$

in the category **Contraction**

This idea yields

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DAGGER CATEGORIES AND THE COMPLEX NUMBERS: AXIOMS FOR THE CATEGORY OF FINITE-DIMENSIONAL HILBERT SPACES AND LINEAR CONTRACTIONS

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This idea yields

- insightful new proofs of the characterisations of the categories of Hilbert spaces
- the first characterisation of a category of finite-dimensional Hilbert spaces

Relevant for quantum computing

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② Categorical Hilbert theory

CATEGORICAL REFORMULATIONS

Theory	Categorical setting
homological algebra	abelian categories
probability theory	Markov categories
differential geometry	tangent categories

Reformulating a theory category theoretically can

- unify and generalise known results,
- reveal new results,
- simplify it, making it more accessible.

CATEGORICAL REFORMULATIONS

Theory	Categorical setting
homological algebra	abelian categories
probability theory	Markov categories
differential geometry	tangent categories
functional analysis	?

R*-categories are a new categorical abstraction of algebraic aspects of Hilbert spaces

R*-CATEGORIES

THE HILBERT-SPACE ANALOGUE OF ABELIAN CATEGORIES

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Abstract. This article introduces R*-categories—an abstraction of categories exhibiting the "algebraic" aspects of the theory of Hilbert spaces. Notably, finite biproducts in R*-categories can be orthogonalised using the Gram–Schmidt process, and generalised notions of positivity and contraction support a variant of Sz.-Nagy's unitary dilation theorem. Underpinning these generalisations is the structure of an involutive identity-on-objects contravariant endofunctor, which encodes adjoints of morphisms. The R*-category axioms are otherwise inspired by those for abelian categories, comprising a few simple properties of products and kernels. Additivity is not assumed, but nevertheless follows. In fact, the similarity with abelian categories runs deeper—R*-categories are quasi-abelian and thus homological. Examples include the category of unitary representations of a group, the category of finite-dimensional inner product modules over a partially ordered division ring, and the category of self-dual Hilbert modules over a W*-algebra.

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M*-categories also include analytic aspects

Articles in preparation

Let C be a $*$-category with

Let C be a ∗-category with

If C = Adjointable then

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(1) a zero object,

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Let $\mathbf{C}$ be a $*$-category with

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(2) binary orthonormal biproducts

Let **C** be a $*$-category with

If **C** = **Adjointable** then

(1) a zero object, — $0 = \{0\}$

(2) binary orthonormal biproducts, — $X \oplus Y = \{(x,y) : x \in X, y \in Y\}$

$\langle (x,y) \mid (u,v) \rangle = \langle x \mid u \rangle + \langle y \mid v \rangle$

Let **C** be a $*$-category with

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(2) binary orthonormal biproducts,

(3) isometric equalisers, such that

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$Eq(f,g) = \{x \in X : fx = gx\}$
with restricted inner product

Let C be a $*$-category with

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  with restricted inner product
- $Ker(f) = Eq(f, 0)$

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$m^* m = 1$

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(5) **Isometry(C)** has directed colimits.

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e.g. $\ell^2(\mathbb{N})$

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R^*-category

(5) Isometry(C) has directed colimits.

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R*-category (1)–(4)

M*-category (1)–(5)

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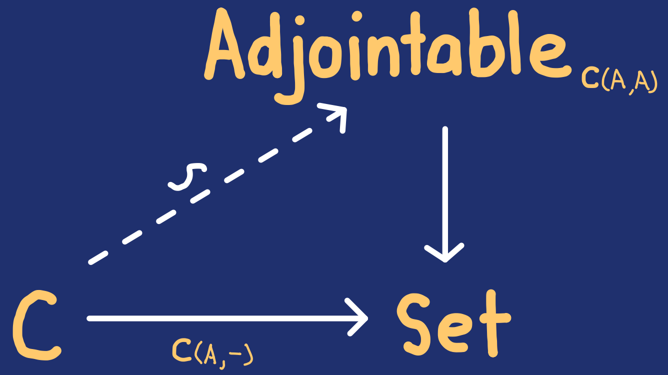
(5) Isometry(C) has directed colimits.

THEOREM:

If C has a simple separator A then C(A,A) is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and



THEORY

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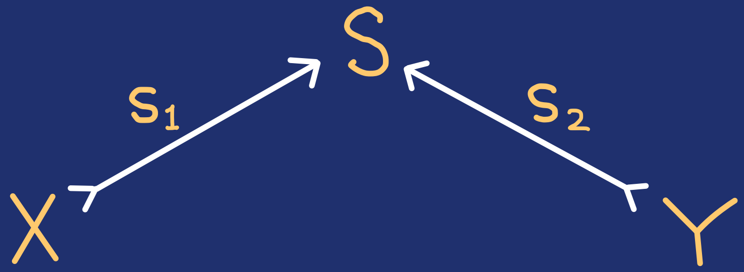
- Canonical partial order and inner products
  $f^*f \geq 0$ $\quad\quad$ $\langle f | g \rangle = f^*g$
- Symmetry (If $a \geq 1$ then a invertible)
- Contractions (Morphisms f with $f^*f \leq 1$)
- Monotone completeness
  (Bounded increasing nets have suprema)

③ Dilators

DEFINITION:

A codilation of $f: X \to Y$ is a cospan of isometries



such that $s_2^* s_1 = f$.

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$$X \overset{s_1}{\rightarrowtail} S \overset{s_2}{\leftarrowtail} Y$$

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A codilator of f is an initial codilation of f.

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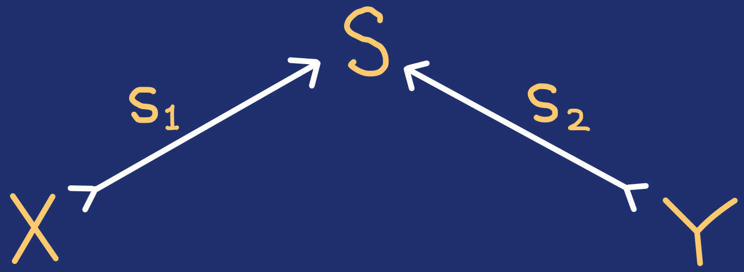

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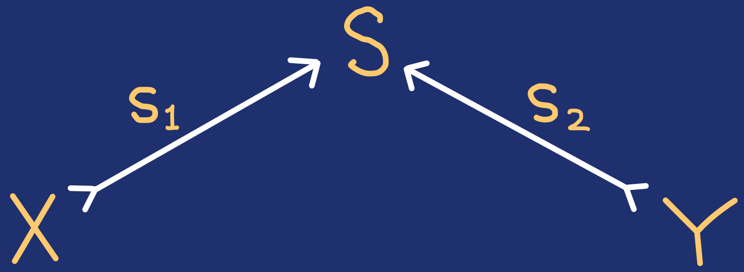

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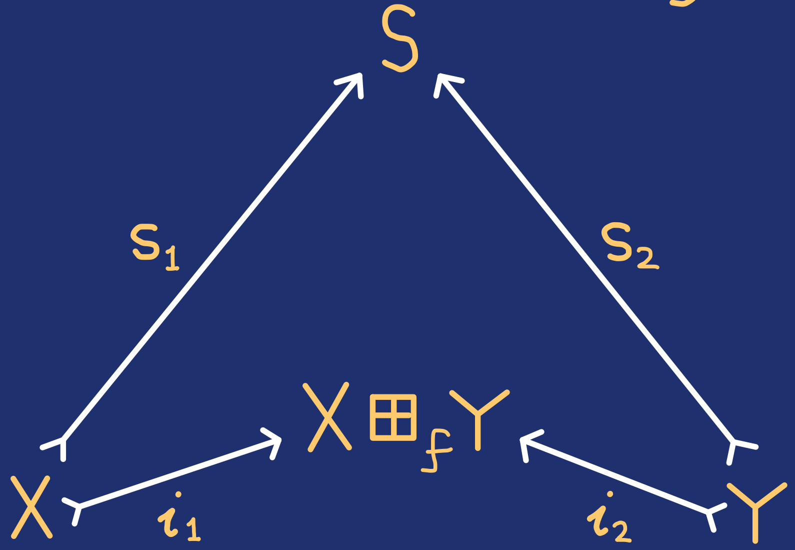
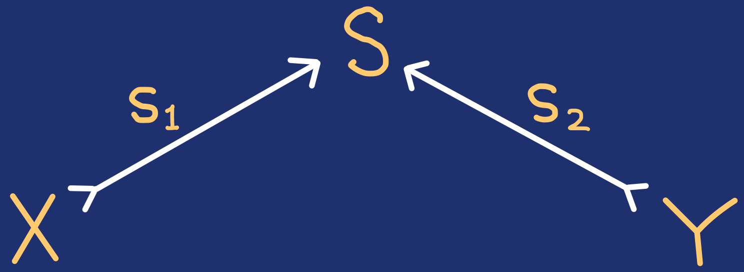


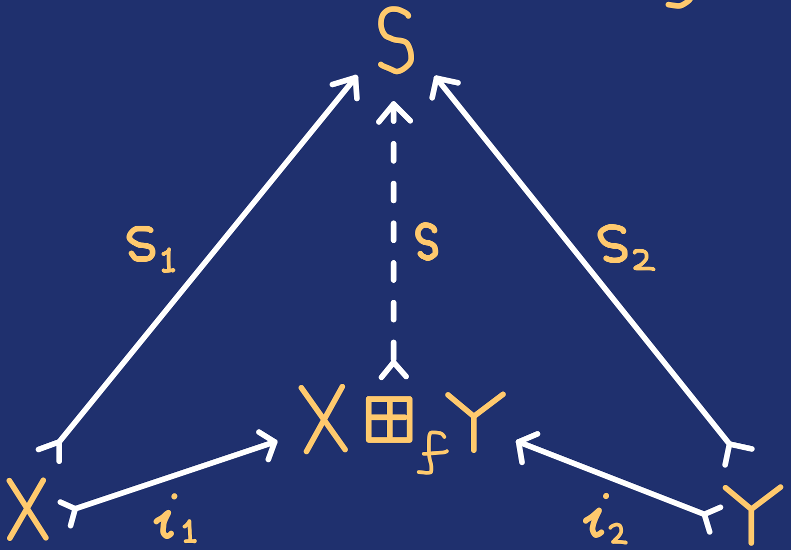
DEFINITION:

A codilation of $f: X \to Y$ is a cospan of isometries



such that $s_2^* s_1 = f$.

A codilator of f is an initial codilation of f.



PROPOSITION:

Every morphism in **Contraction** has a codilator

$$X \xrightarrow{f} Y$$

Inspired by minimal unitary dilations

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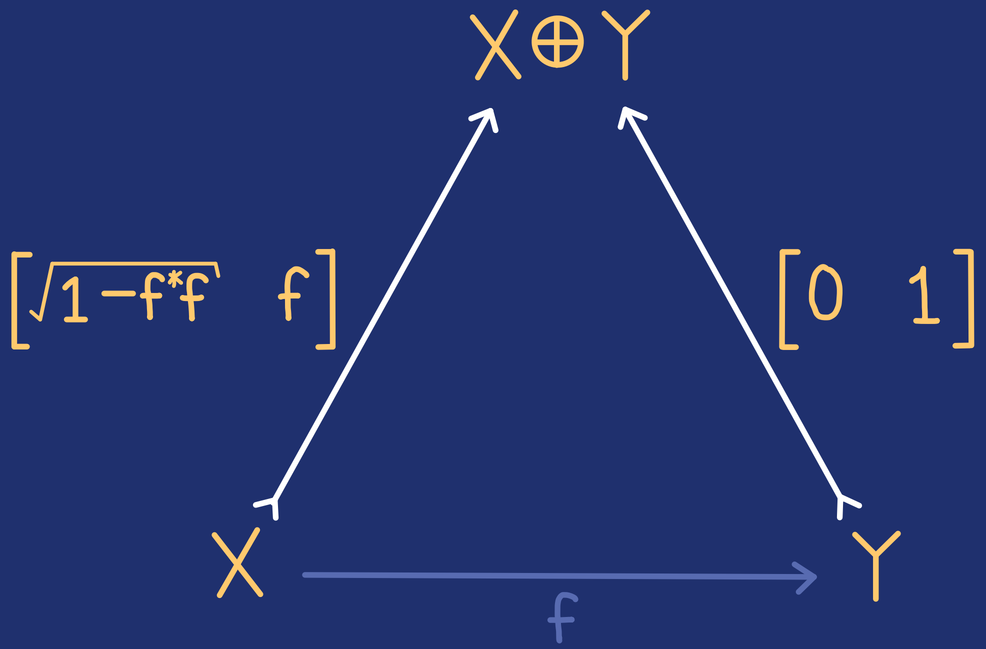

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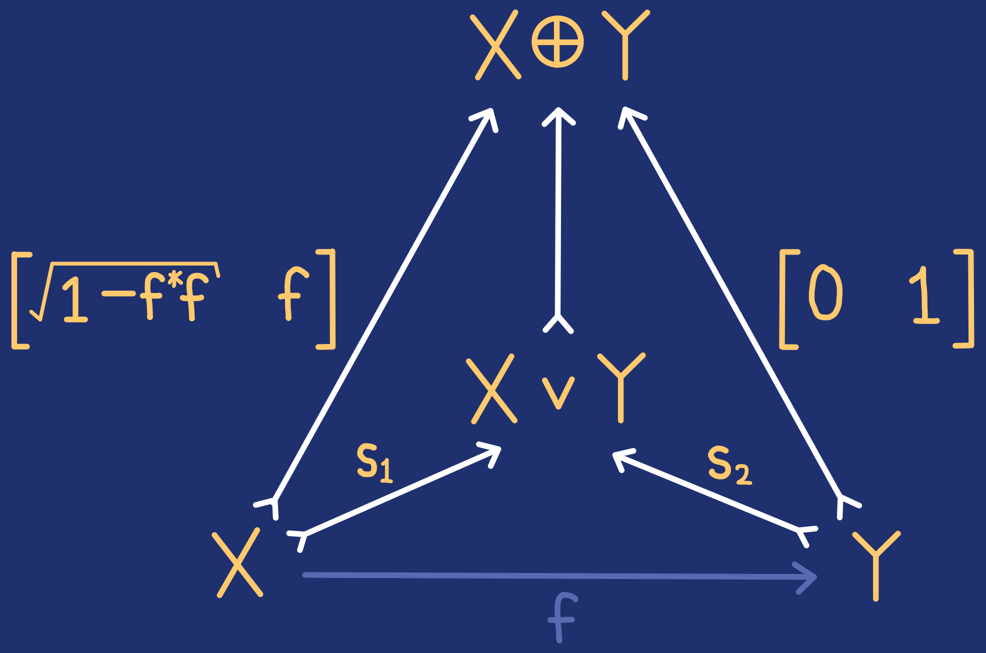

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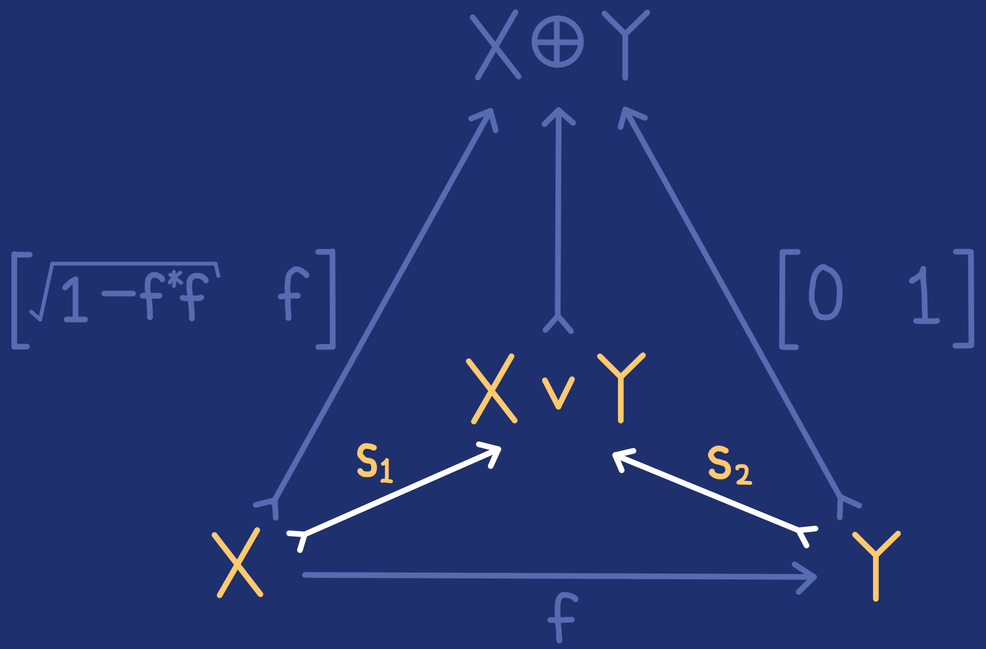

Inspired by minimal unitary dilations

Other $*$-categories with codilators of all morphisms:

- Sets and partial bijections
- Sets and bitotal relations
- Finite probabity spaces and stochastic maps
  (f^* is the Bayesian inverse of f)

Universal property for $\oplus$ in Contraction

Monotone completeness of M*-categories

Dilators are useful

Louis Lemonnier
Semantics for symmetric pattern matching

Monotone completeness of M*-categories

Universal property for ⊕ in Contraction

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PROPOSITION:

Isometry → Contraction preserves directed colimits

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PROOF:

$$X_1 \rightarrowtail X_2 \rightarrowtail \cdots$$

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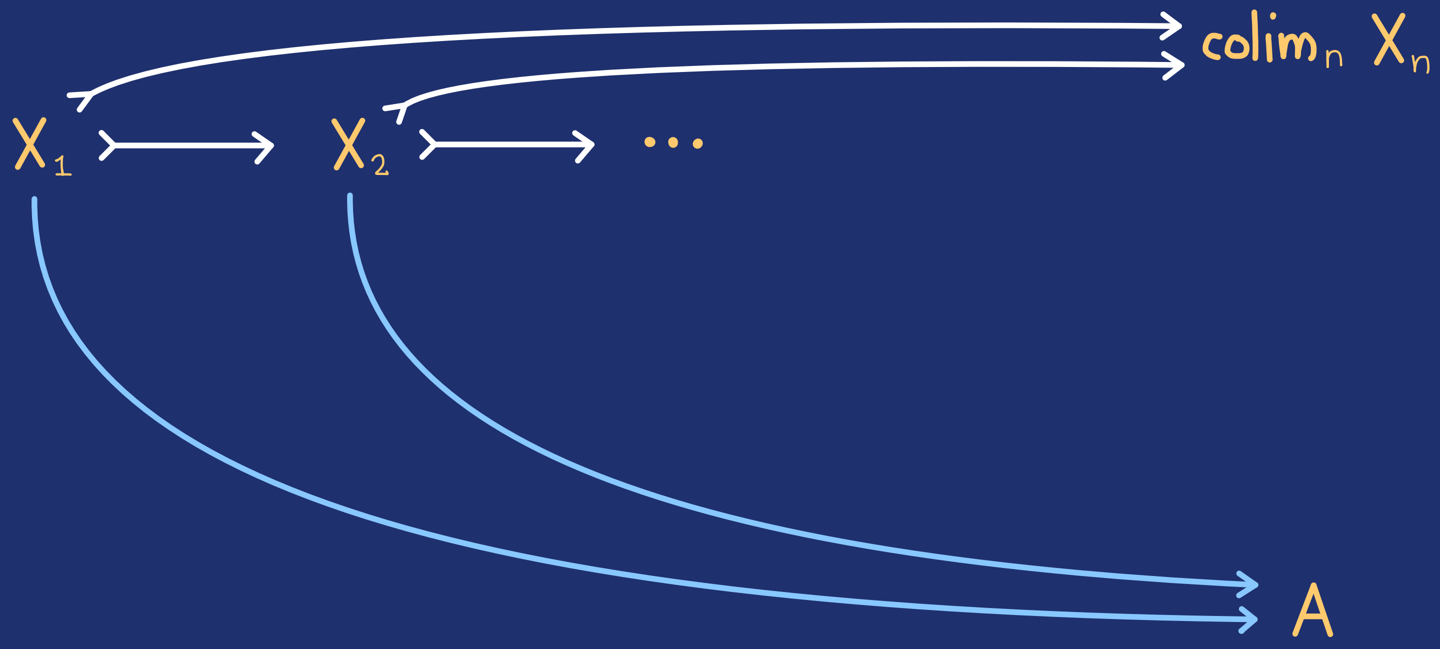

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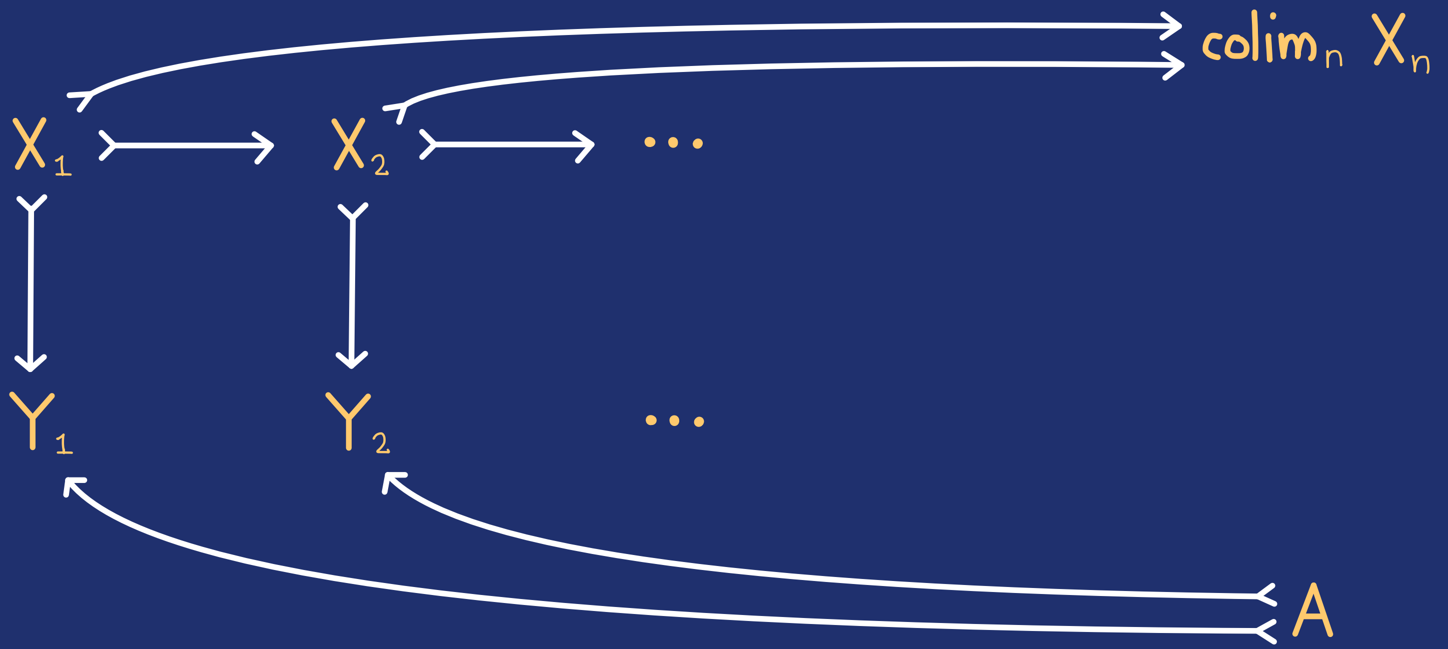

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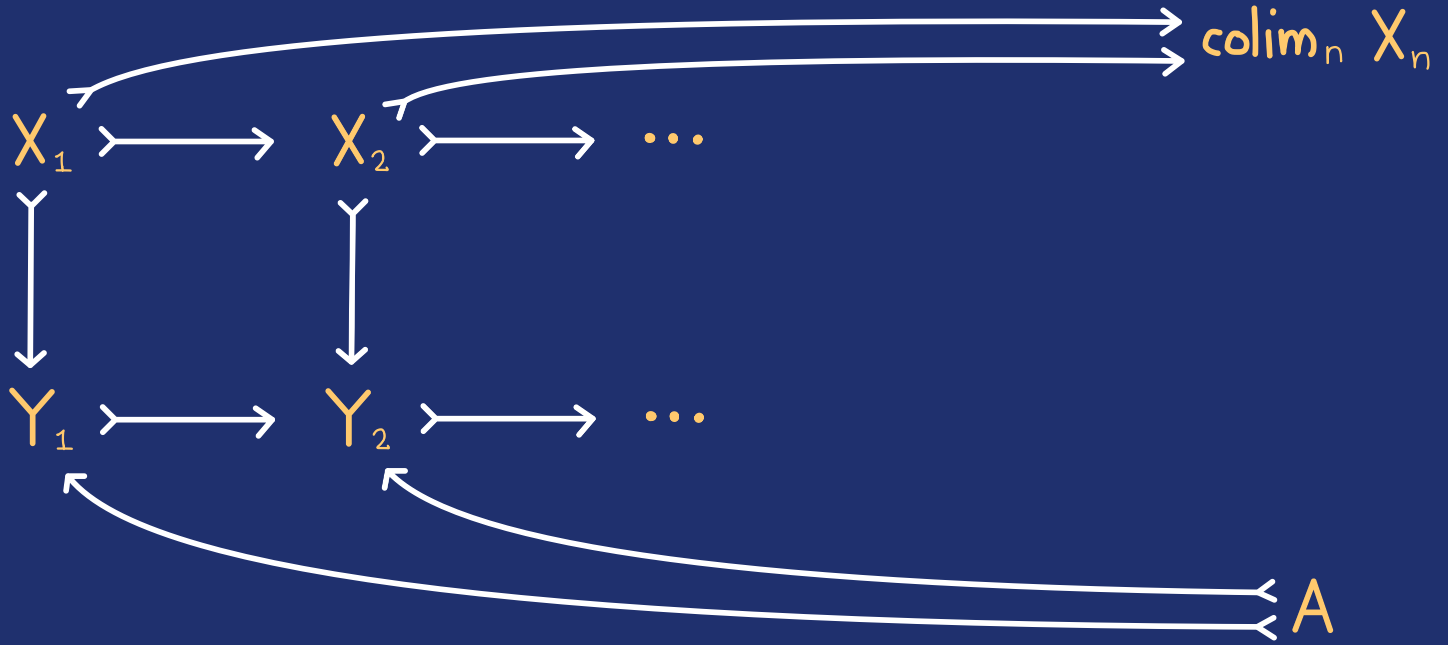

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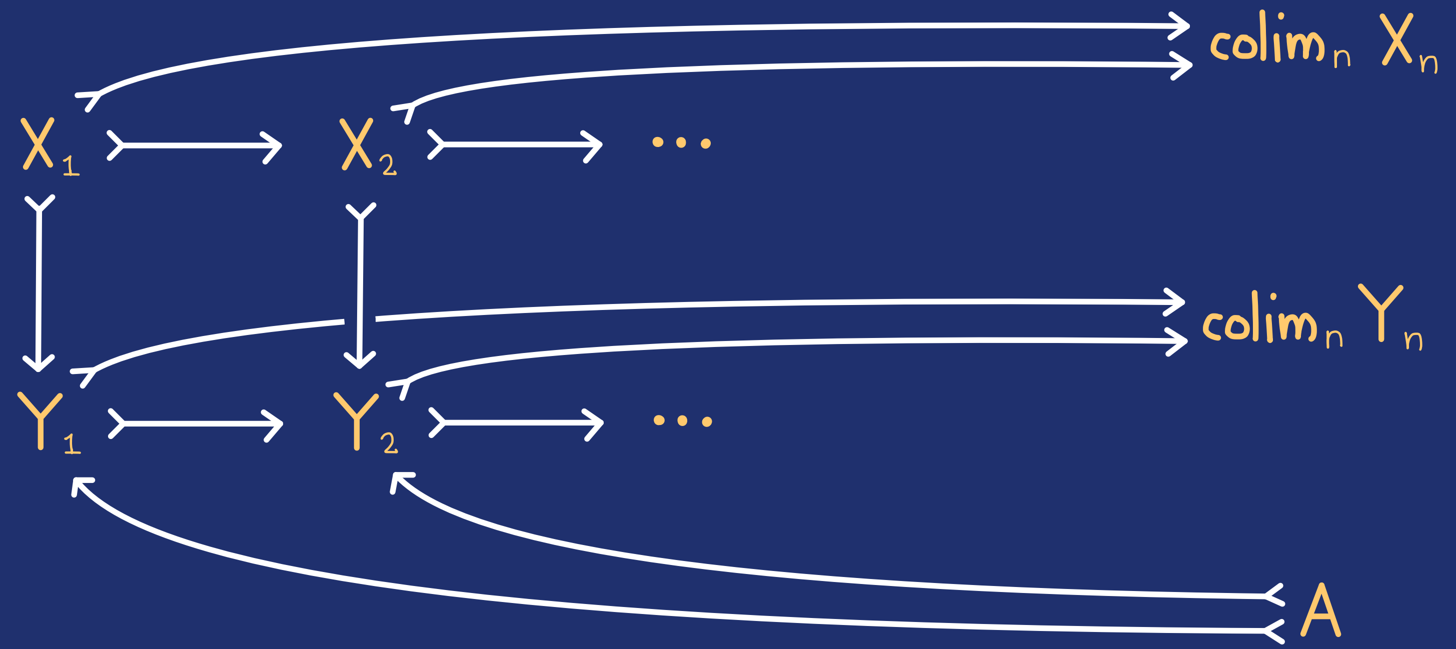

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Isometry → Contraction preserves directed colimits

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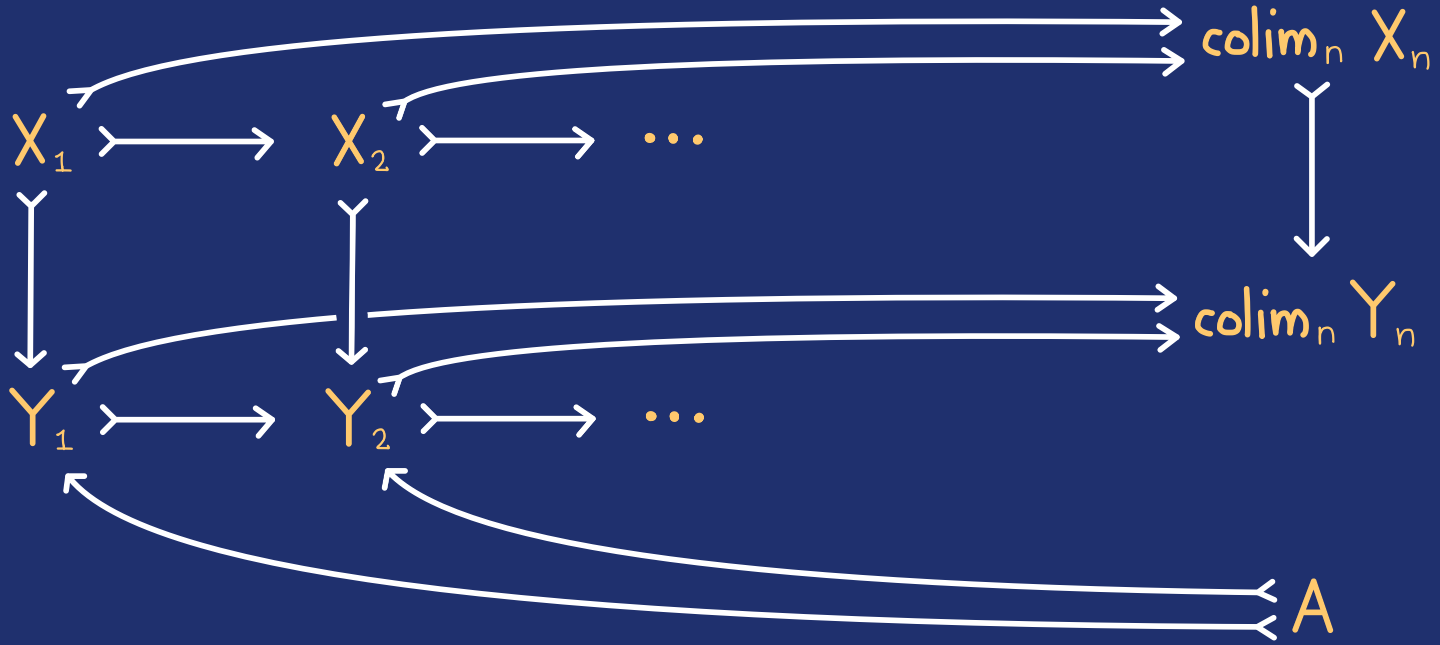
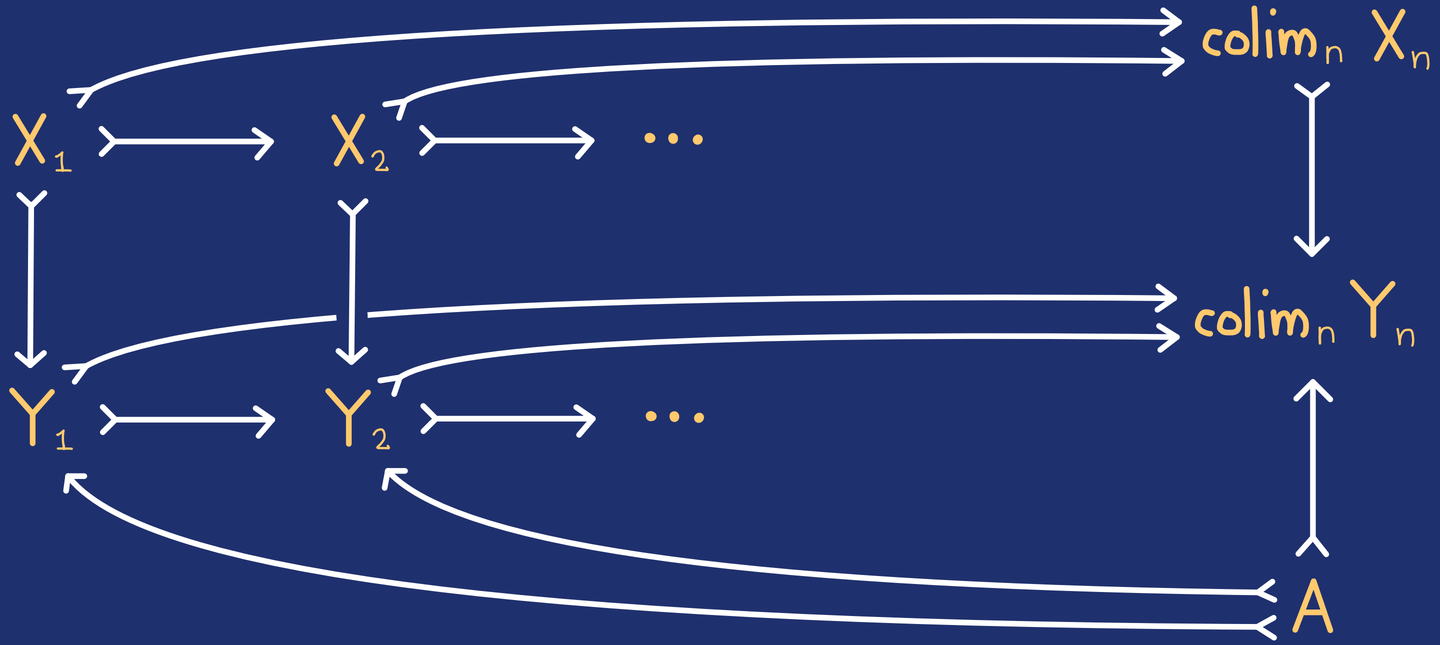


PROPOSITION:

Isometry $\to$ Contraction preserves directed colimits

PROOF:



SUMMARY

- Limits are limits
- R*- and M*-categories for Hilbert theory
- Dilators relate isometries and contractions

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NEXT STEPS

- Contractions as relations in **Isometry**

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- Contractions as relations in **Isometry**
- Axioms for **Isometry**

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- Limits are limits
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NEXT STEPS

- Contractions as relations in **Isometry**
- Axioms for **Isometry**
- Axioms for a category of probability spaces

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Reformulating a theory category theoretically can

- unify and generalise known results,

  e.g. Discrete and continuous cases of the Blackwood-Sherman-Stein theorem on statistical experiments unified for the first time via Markov categories

- reveal new results,

  e.g. convergence in mean for backward martingales indexed by an arbitrary net was proved for the first time (according to the authors) using dagger categories

- simplify it, making it more accessible.

  e.g. to theoretical computer scientists who already know category theory

PROPOSITION: Every morphism in FinPS has a dilator

bloom-shriek factorisation
"The information loss of a stochastic map"

coisometric
⇔
deterministic