Enriched Bisimulations

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(Joint work with Bryce Clarke)

Proofs and Algorithms Seminar
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Propositional Modal Logic

Syntax

$\Box$

necessarily/
always

$\Diamond$

possibly/
sometimes

modal formulae

$$\Diamond \mathit{Eat} \rightarrow \neg \Box \mathit{Hungry}$$

atomic formulae

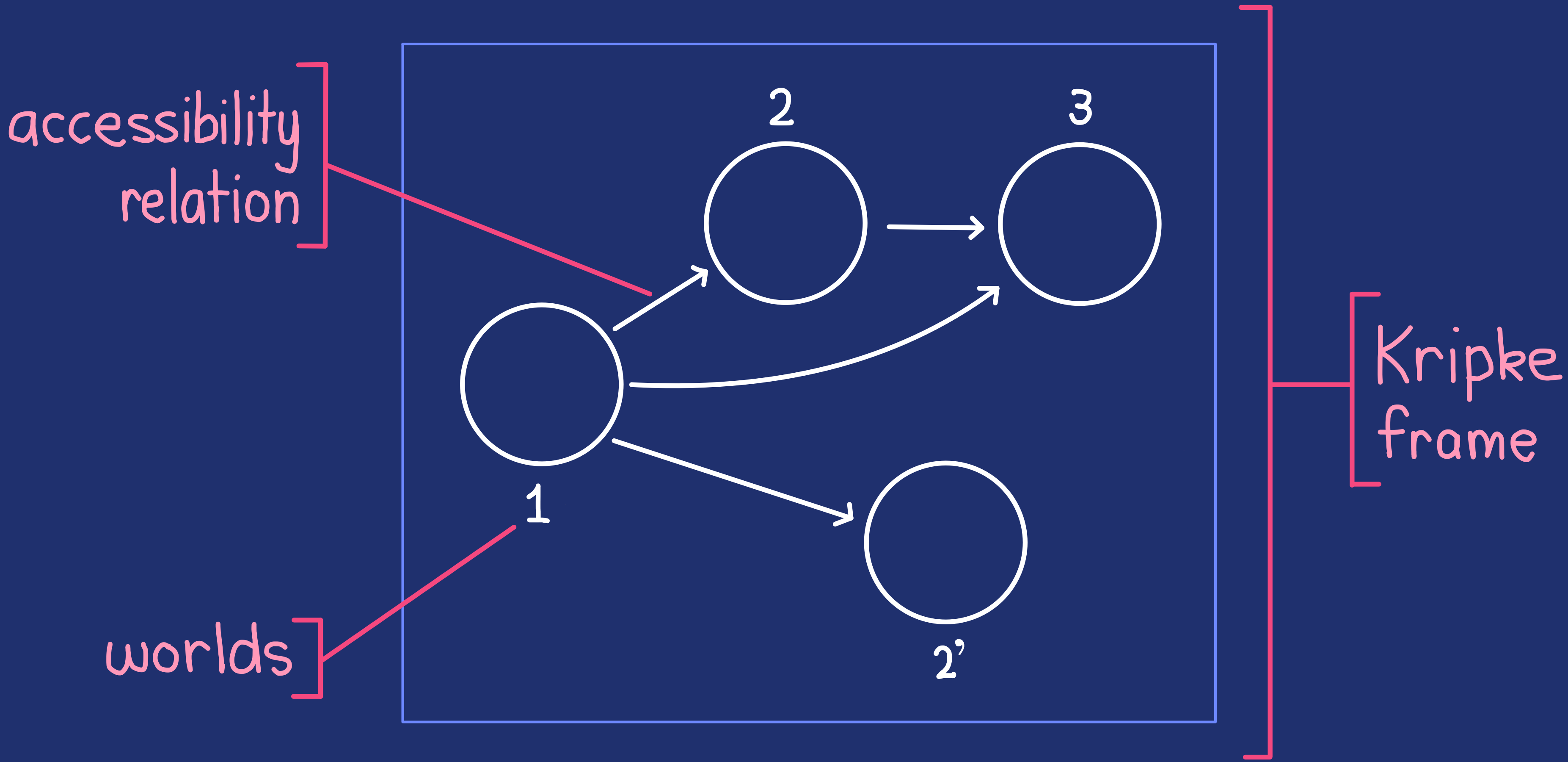
modal operators

$\Box$ $\Diamond$

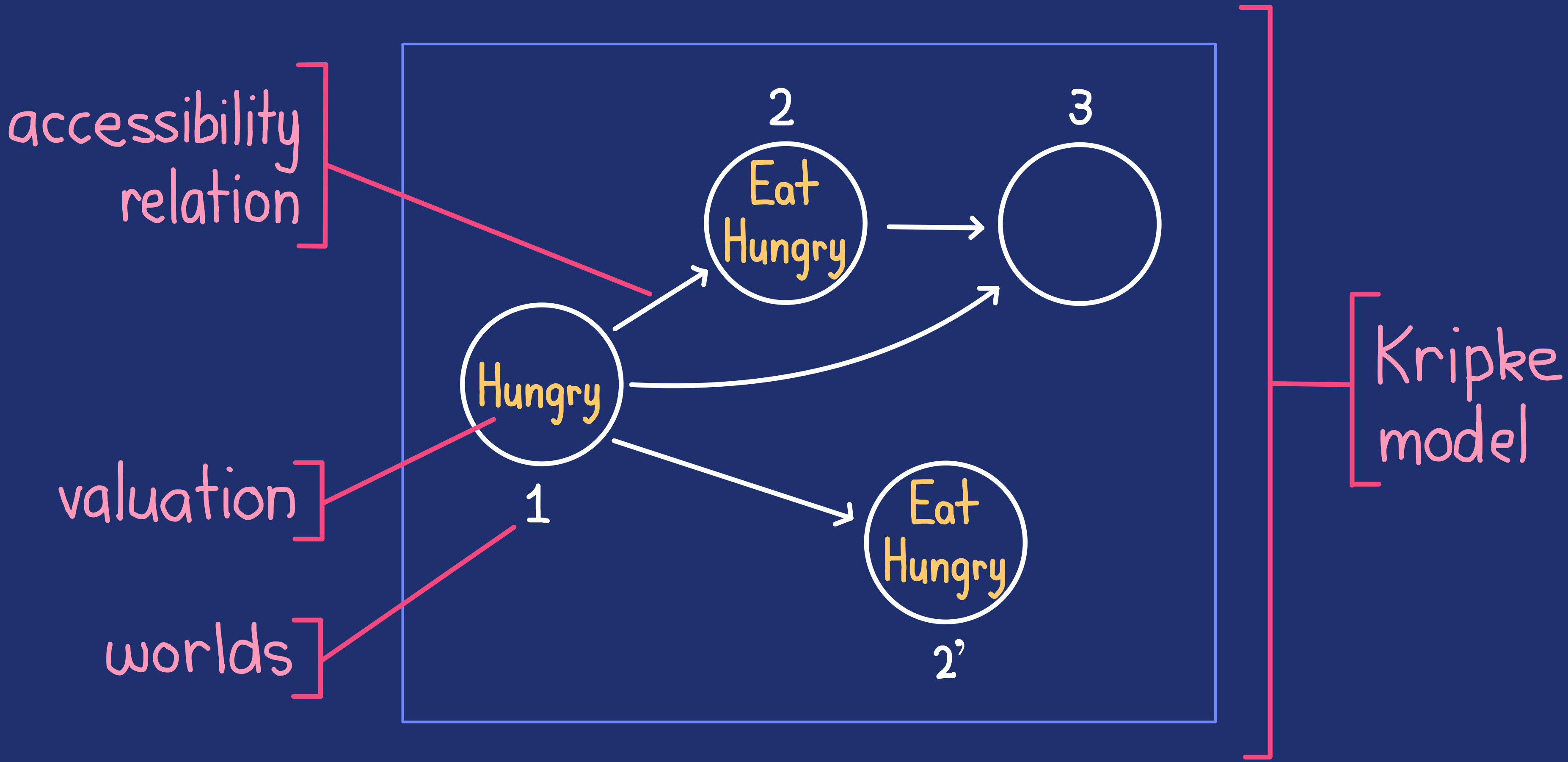
propositional operators

$\wedge$ $\vee$ $\top$ $\bot$ $\rightarrow$ ⌐

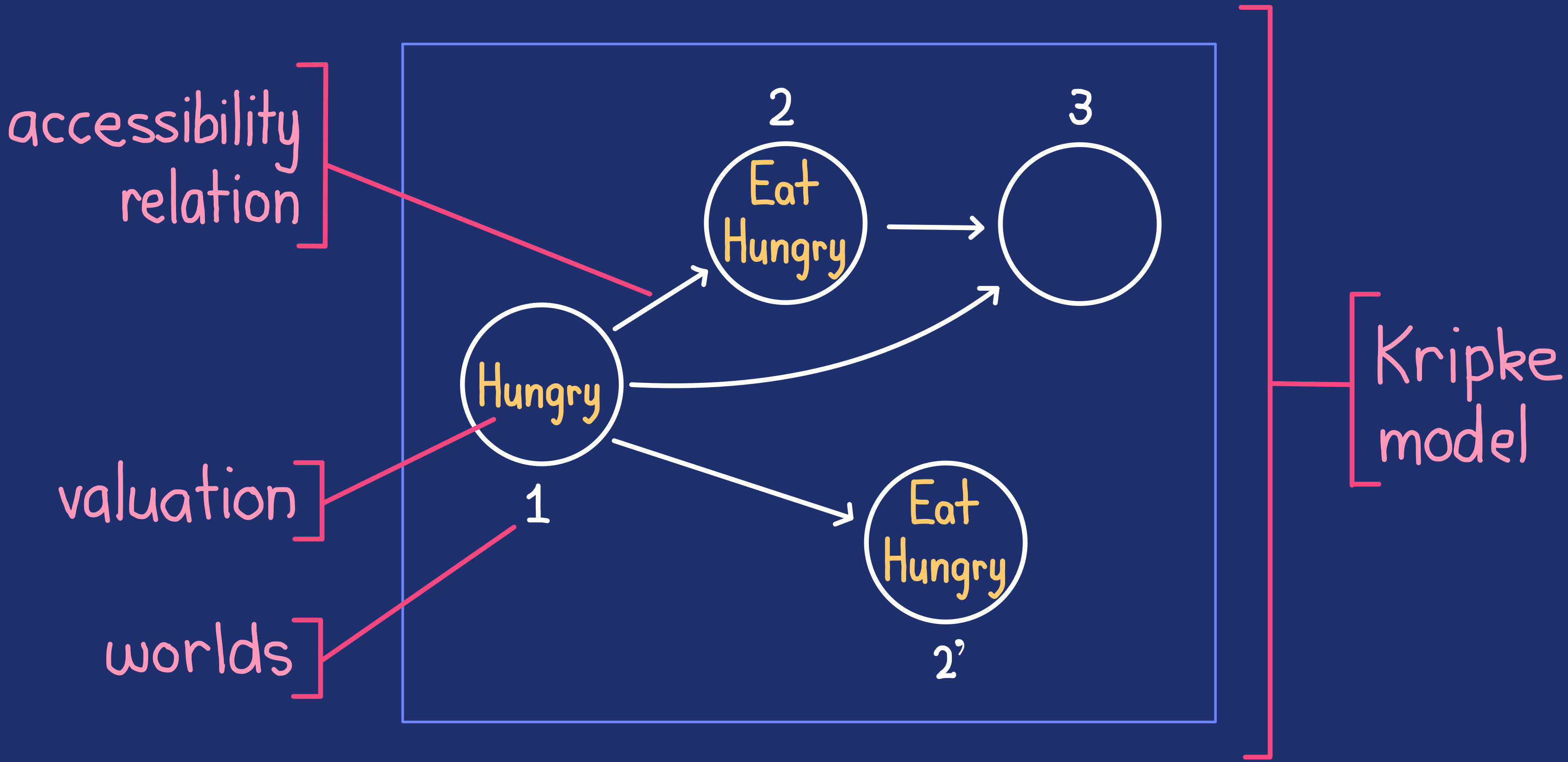
Semantics



Semantics


accessibility relation
valuation
worlds
2
3
Eat
Hungry
Hungry
1
Eat
Hungry
2'
Kripke model

SEMANTICS



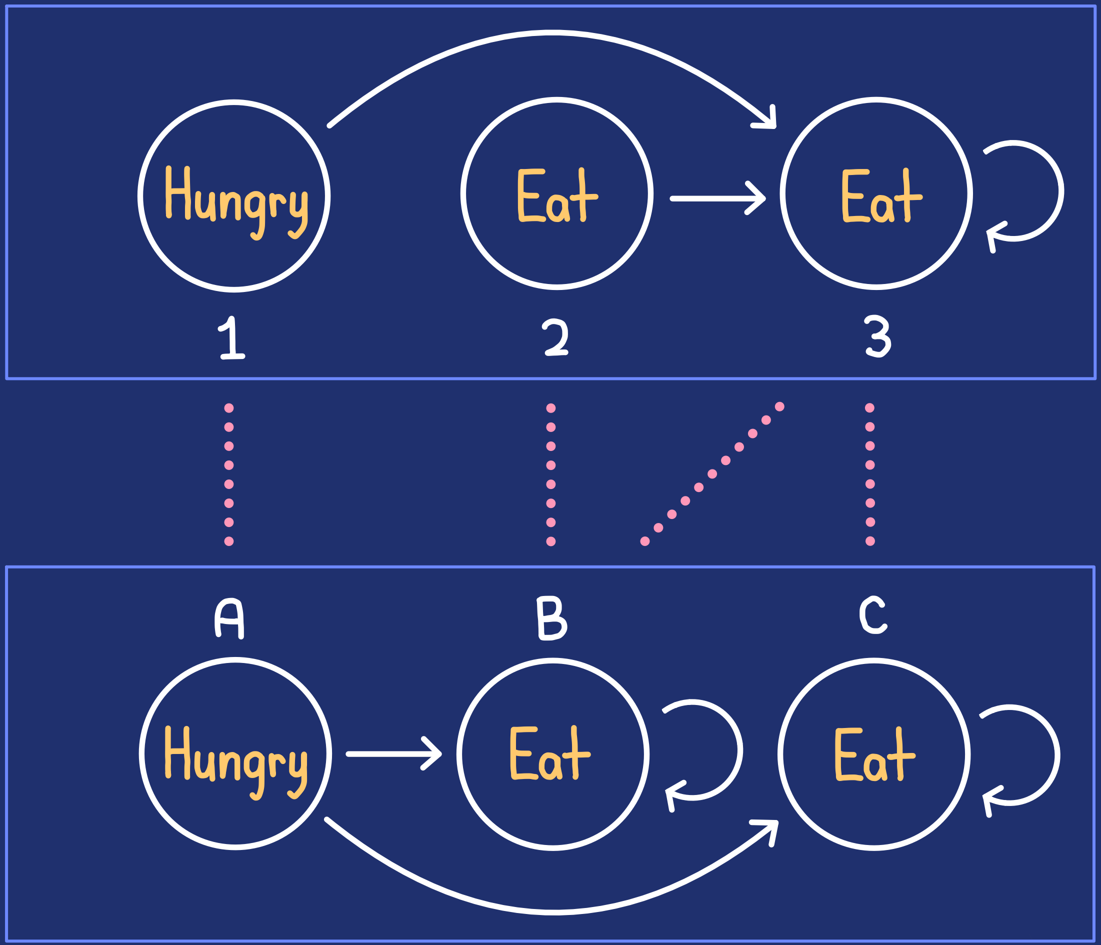
$\Diamond \text{Eat} \rightarrow \neg \Box \text{Hungry}$ holds in world 1 but not world 2'

Which worlds satisfy
the same modal formulae?

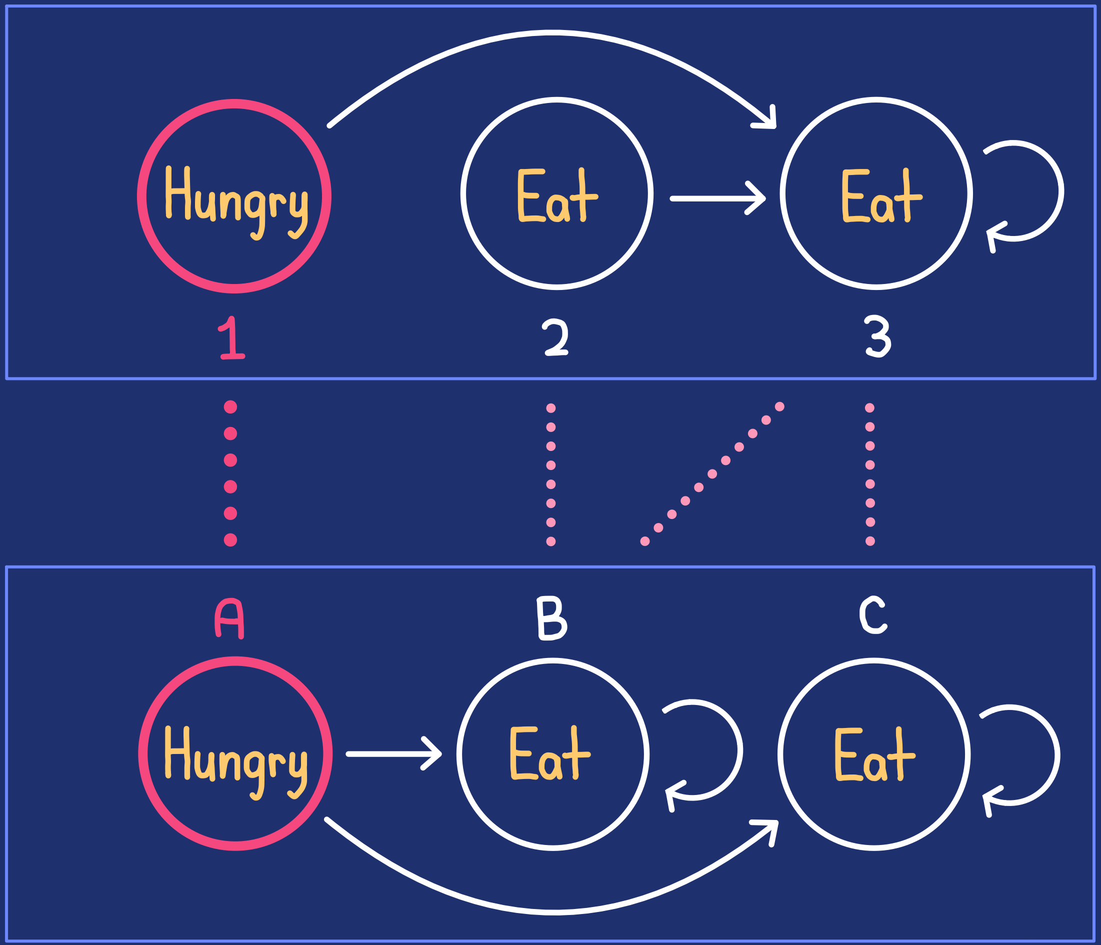
Those related by a bisimulation.*
(or p-relation or zig-zag relation)

*assuming finitely many worlds

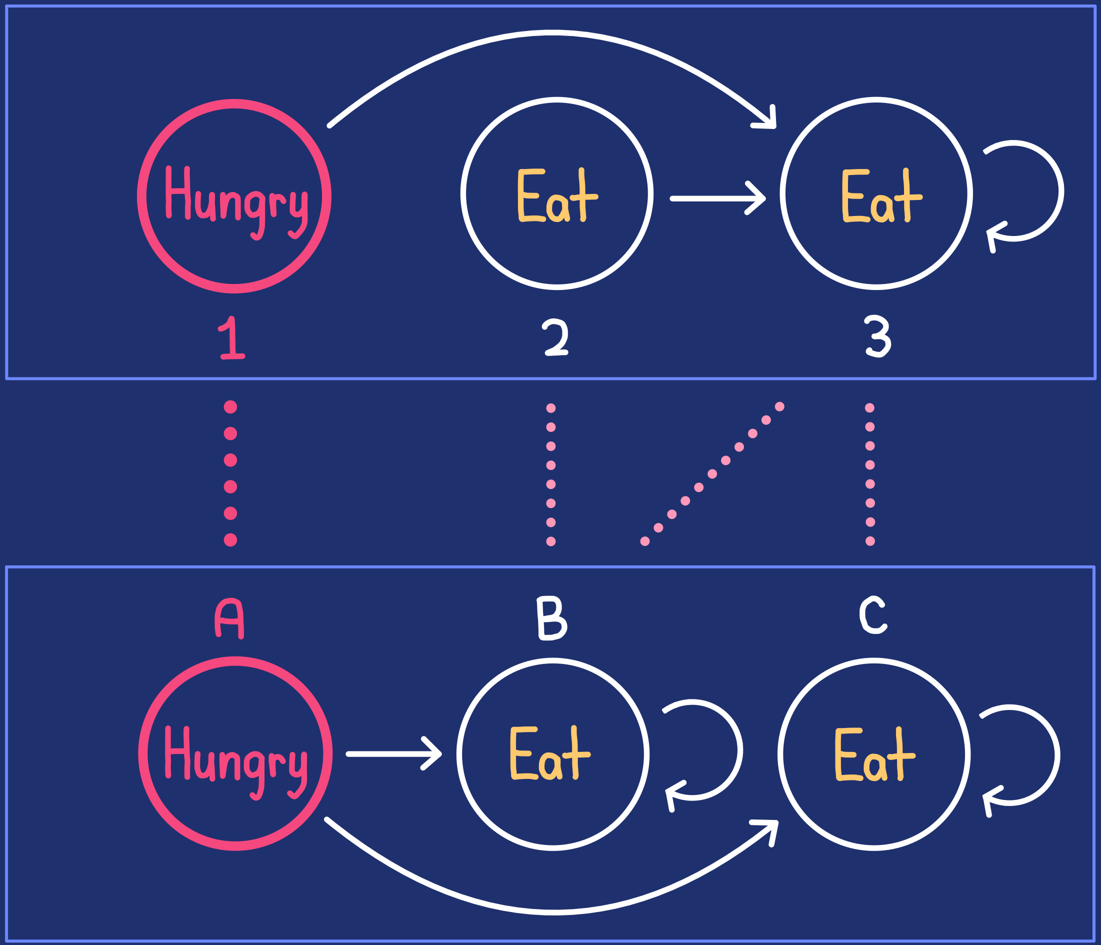
Bisimulation



Bisimulation



Bisimulation



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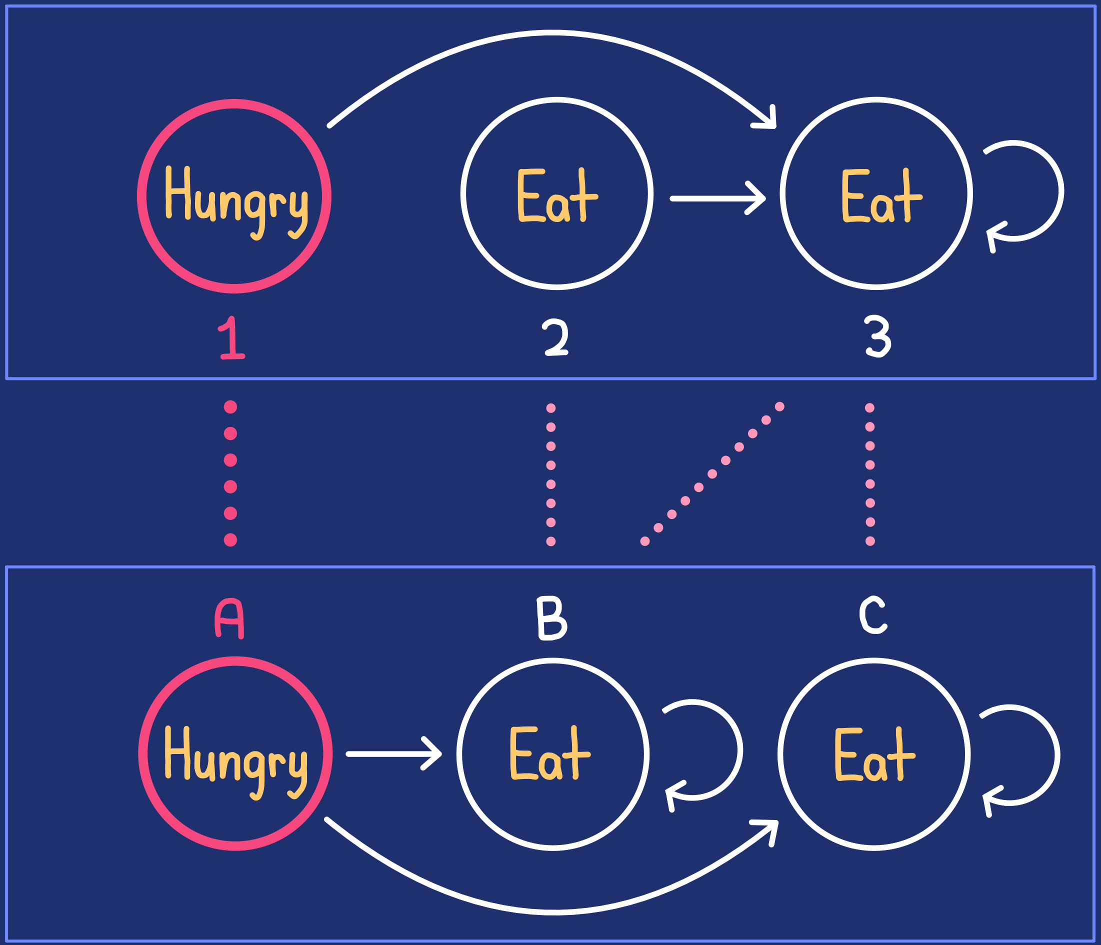
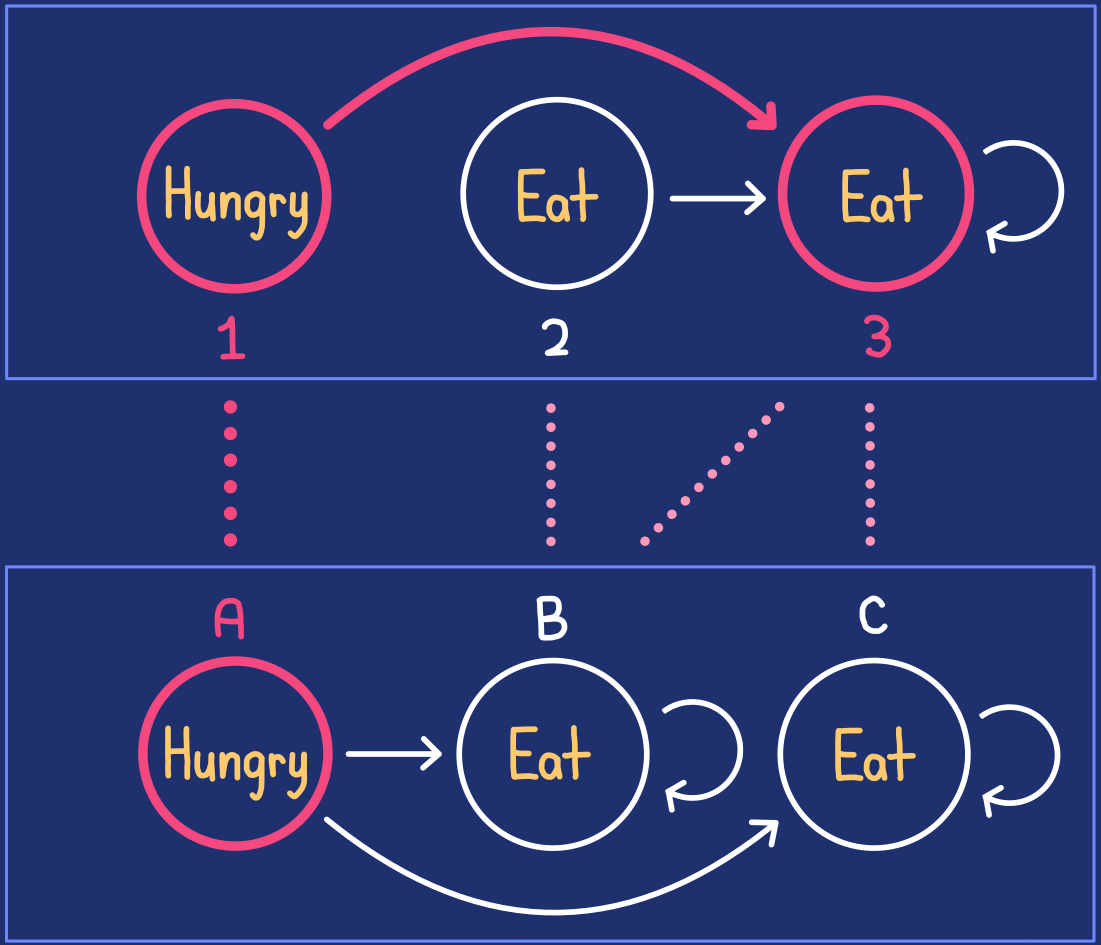
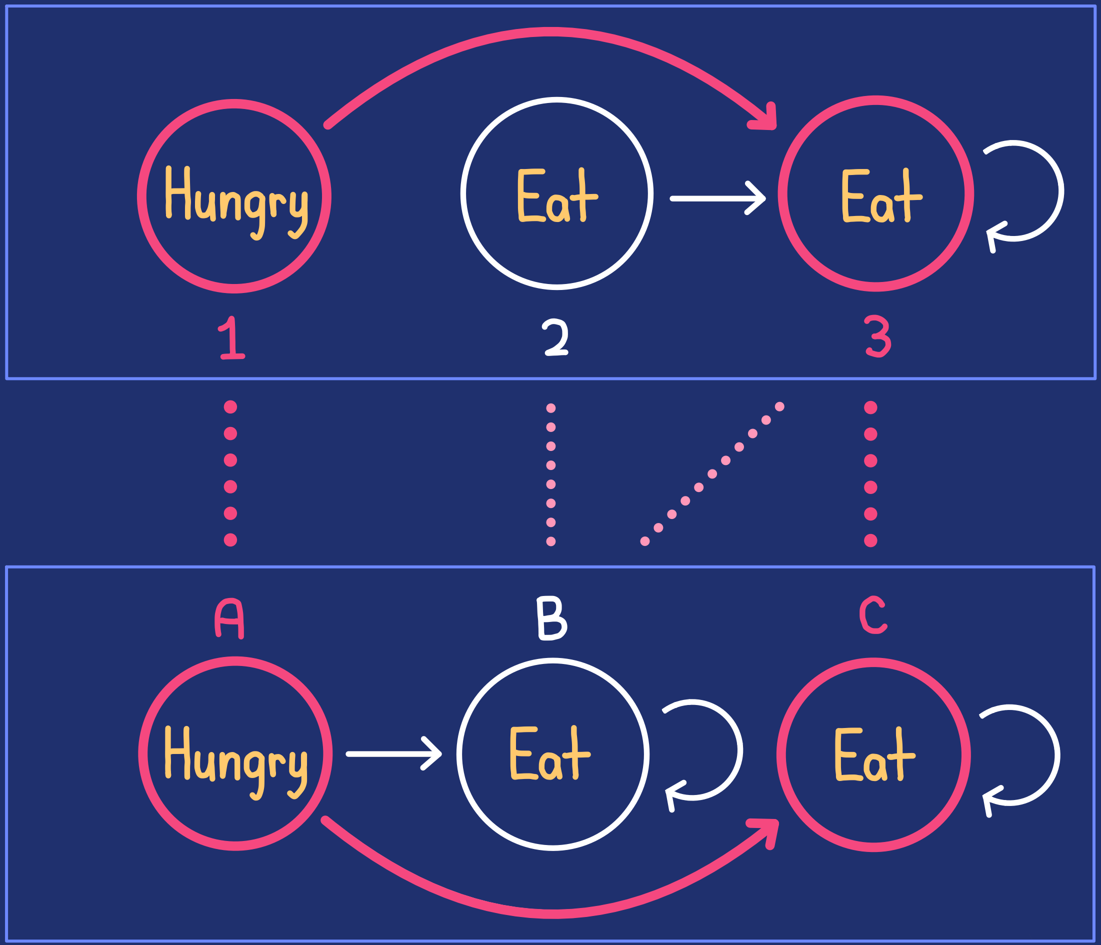


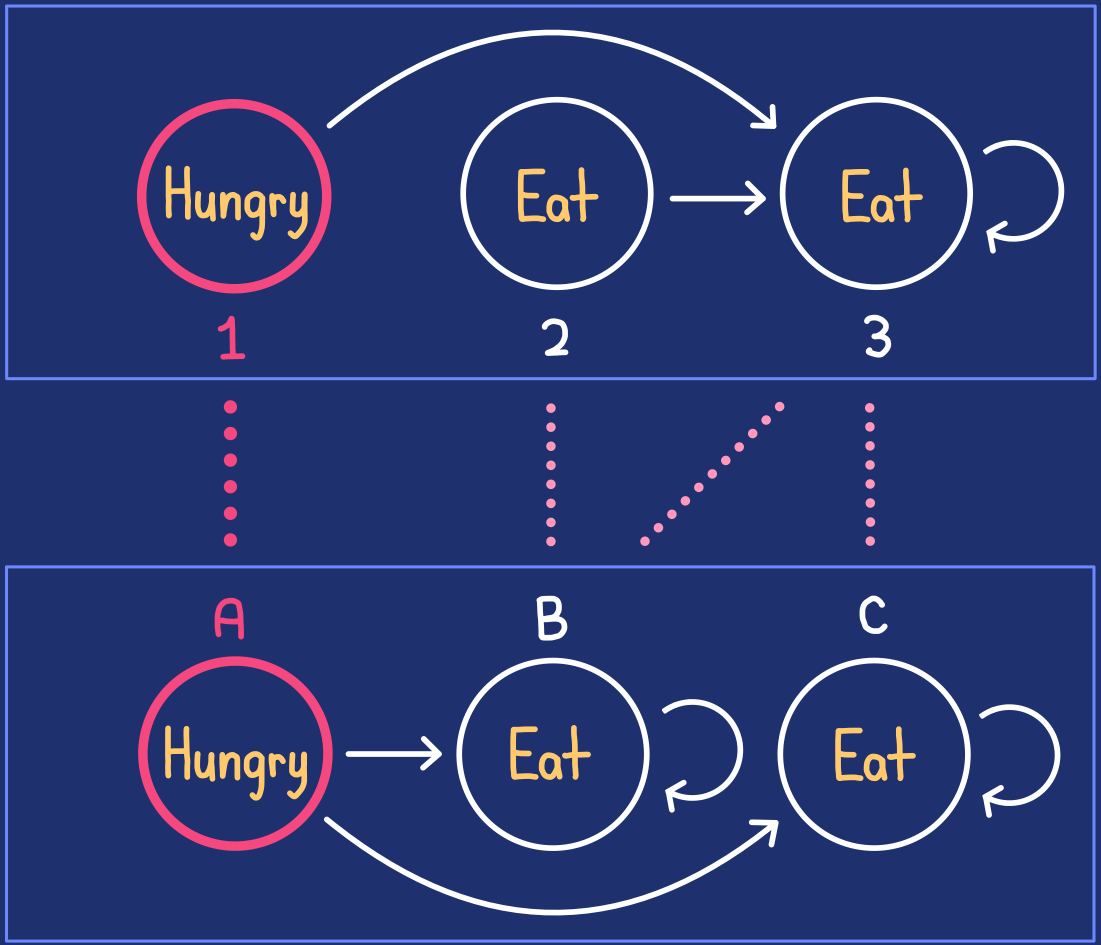
Bisimulation



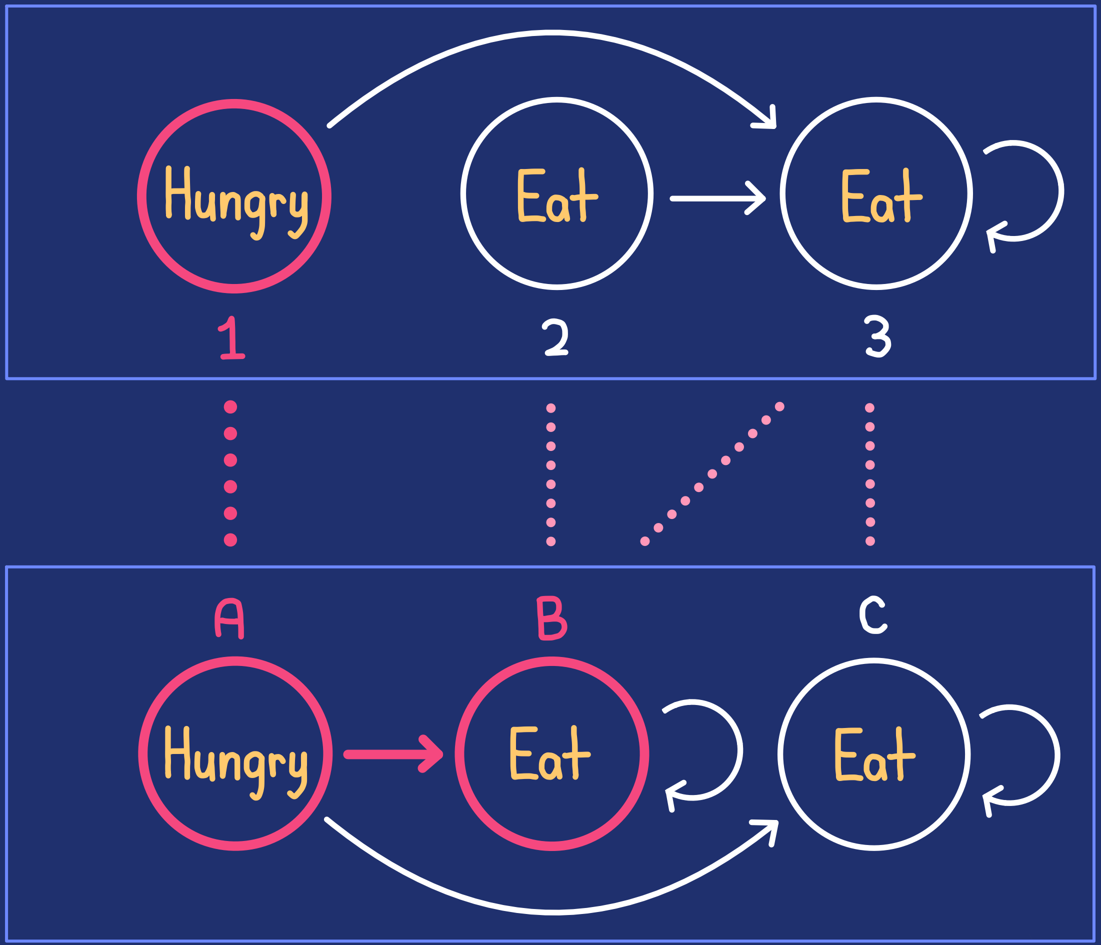
Bisimulation



Bisimulation



Bisimulation



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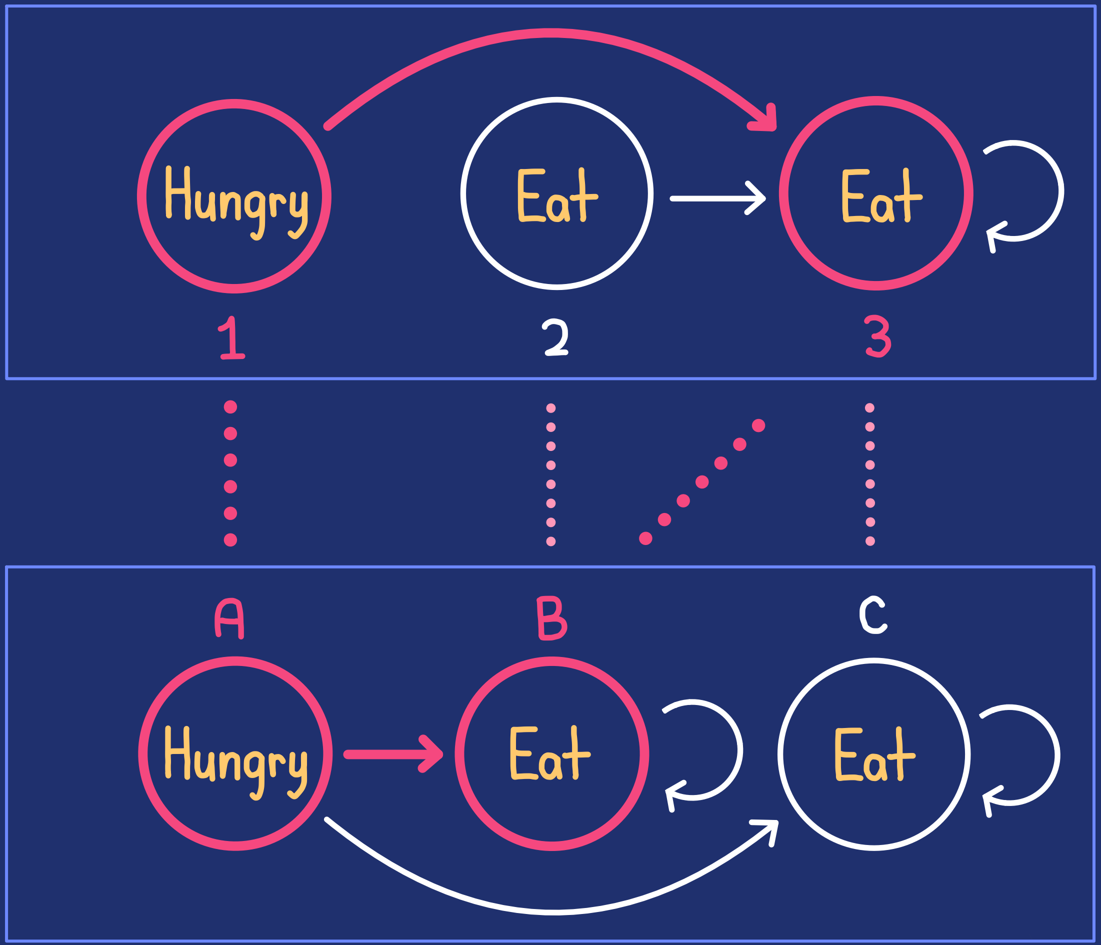
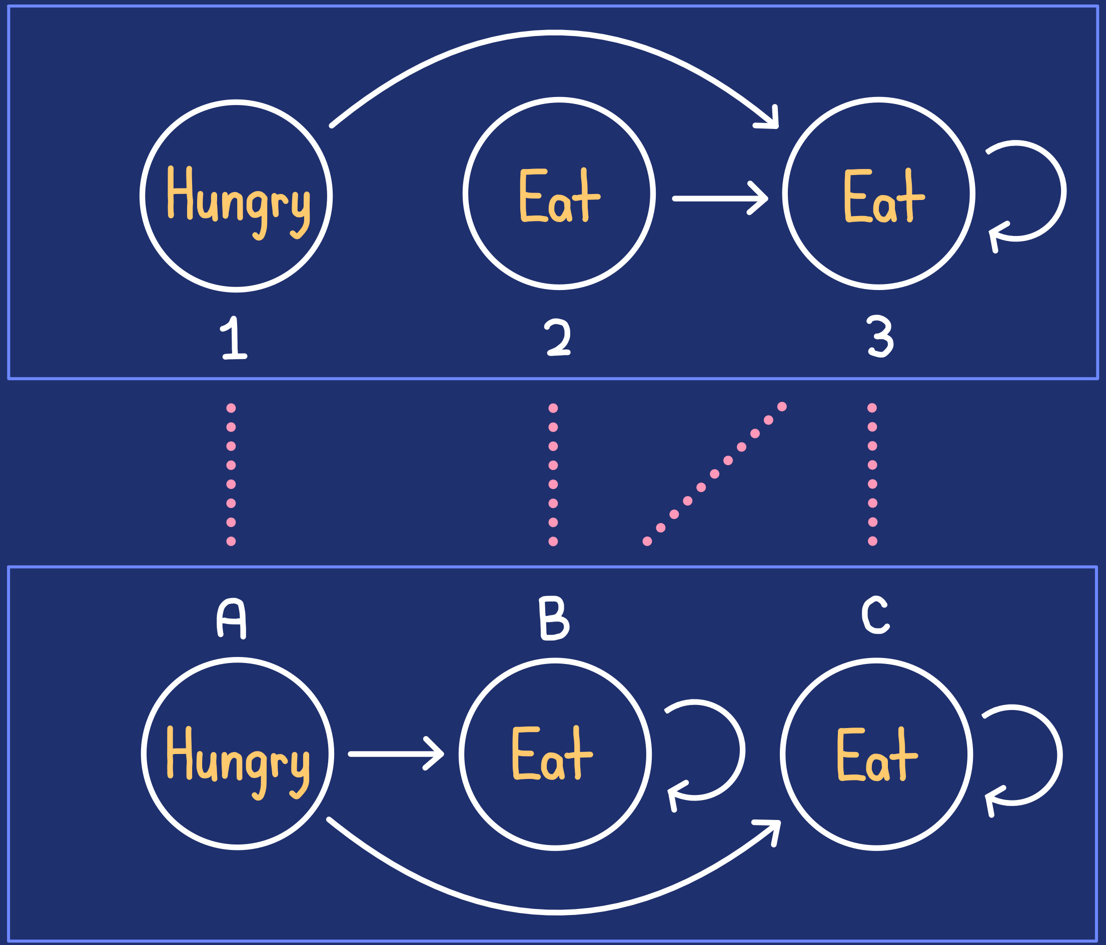
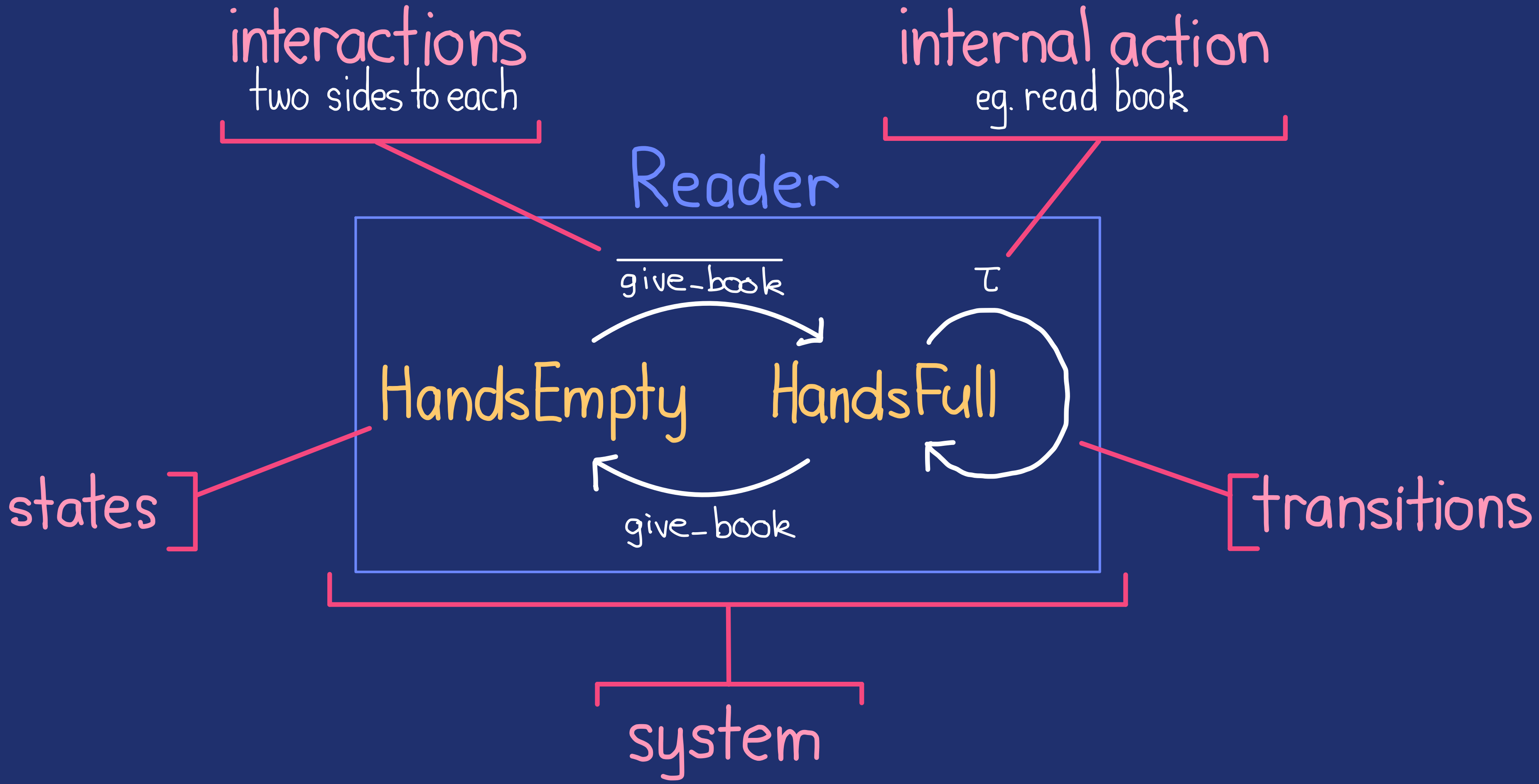


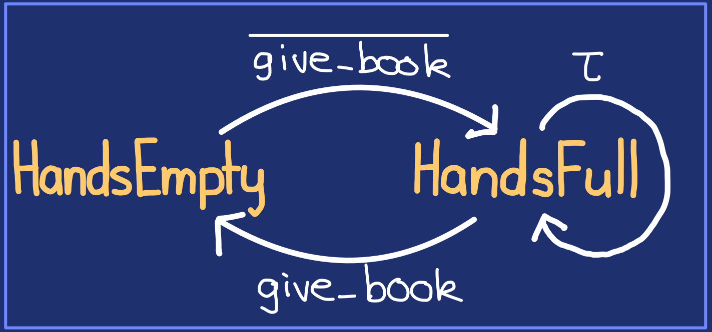
Bisimulation

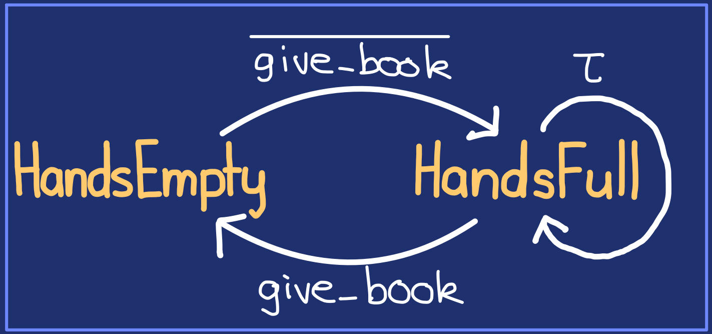


Communication and Concurrency


interactions
two sides to each
internal action
eg. read book
Reader
$\overline{\text{give\_book}}$
τ
HandsEmpty
HandsFull
give_book
states
transitions
system

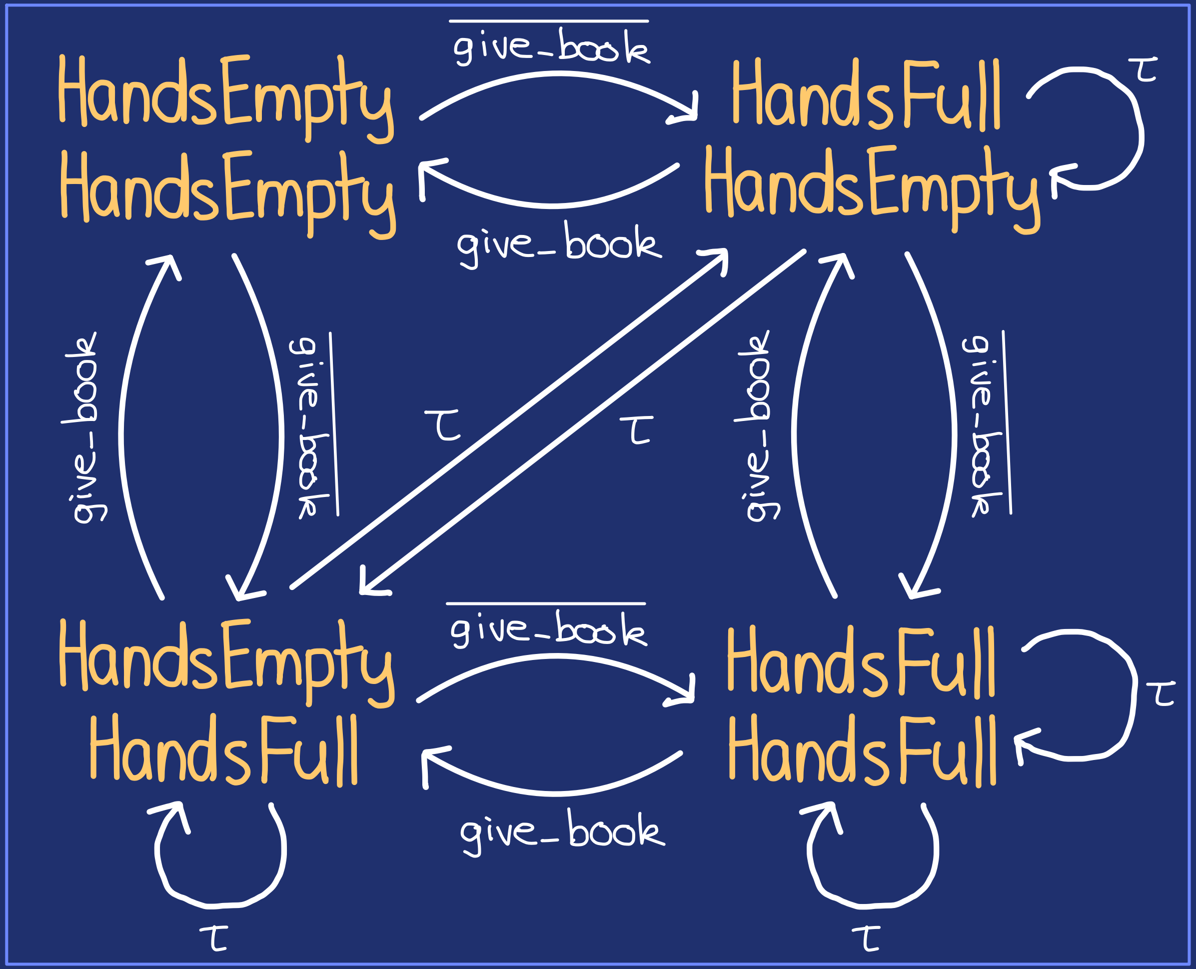
Reader1



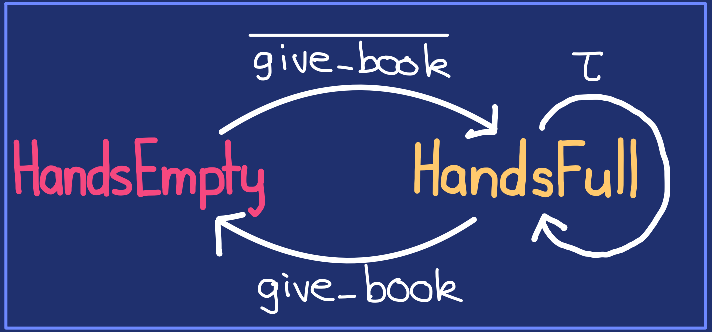


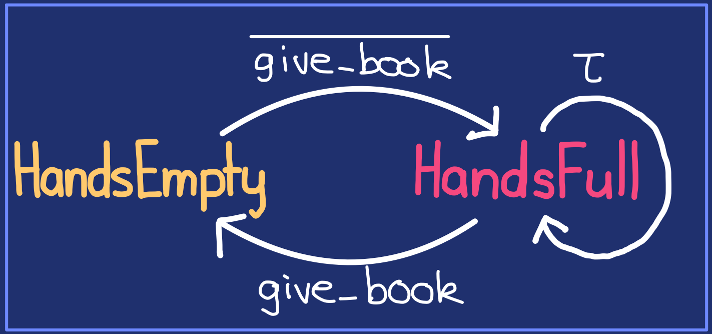
Reader2

ReaderPair



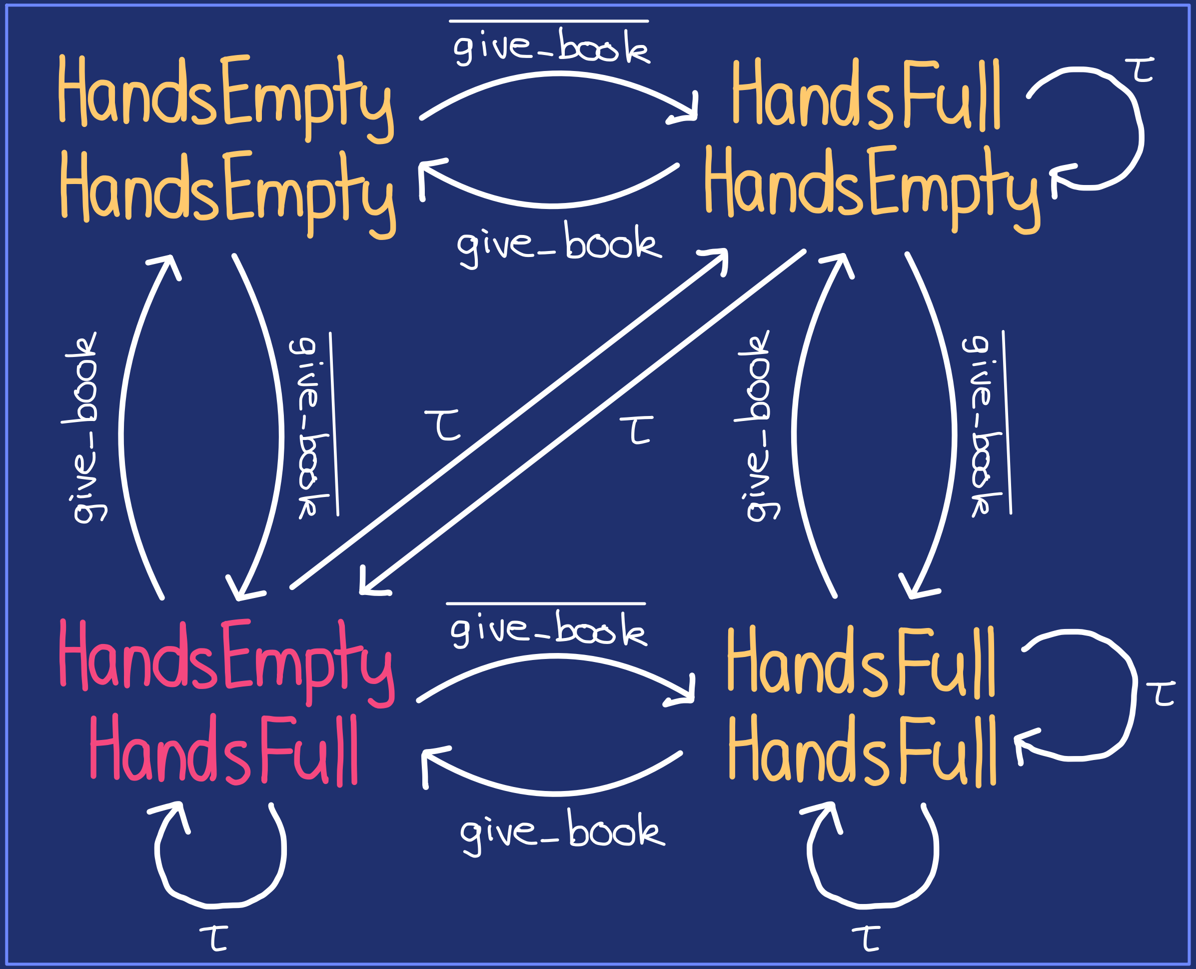
Reader1



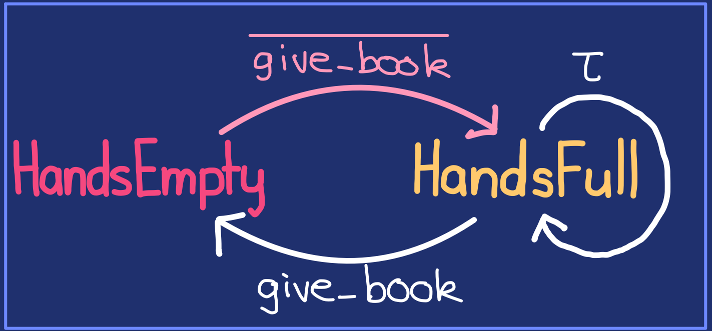


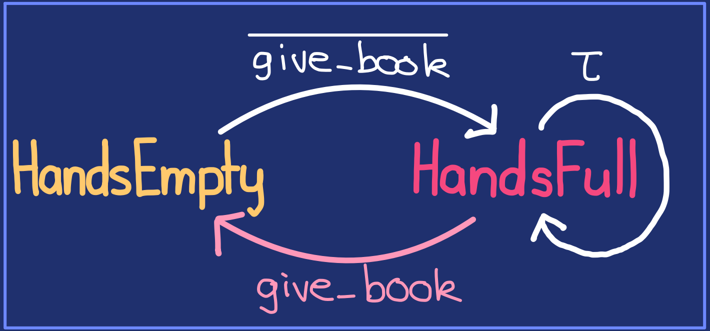
Reader2

ReaderPair



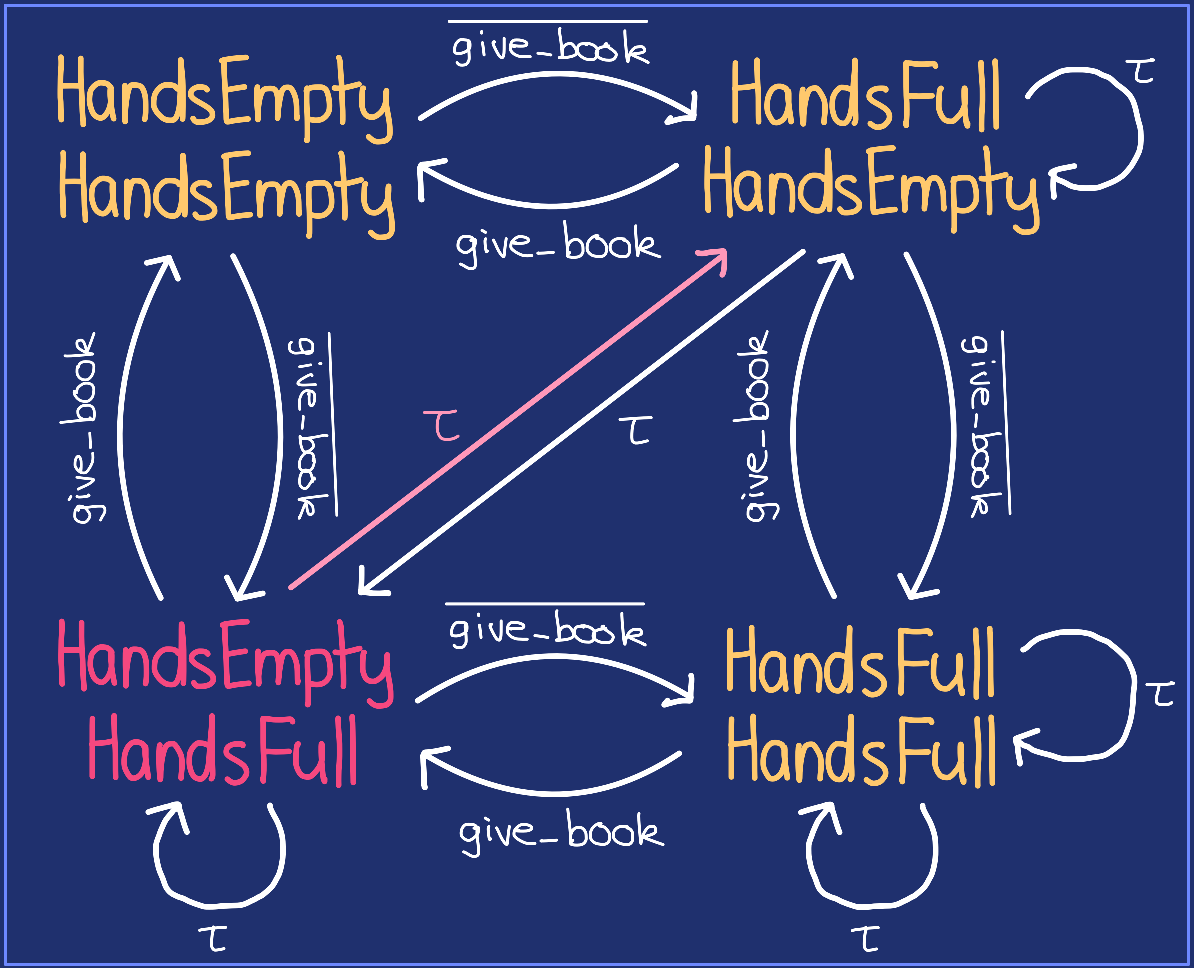
Reader1



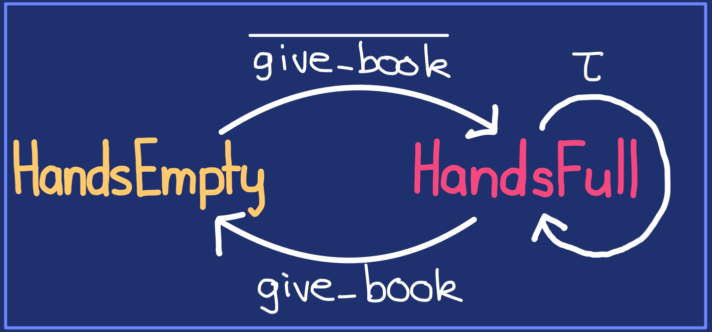


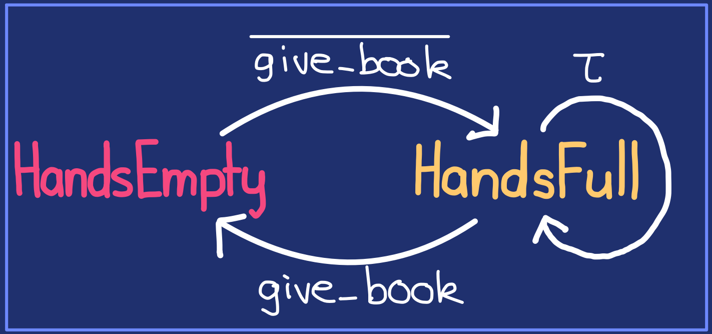
Reader2

ReaderPair



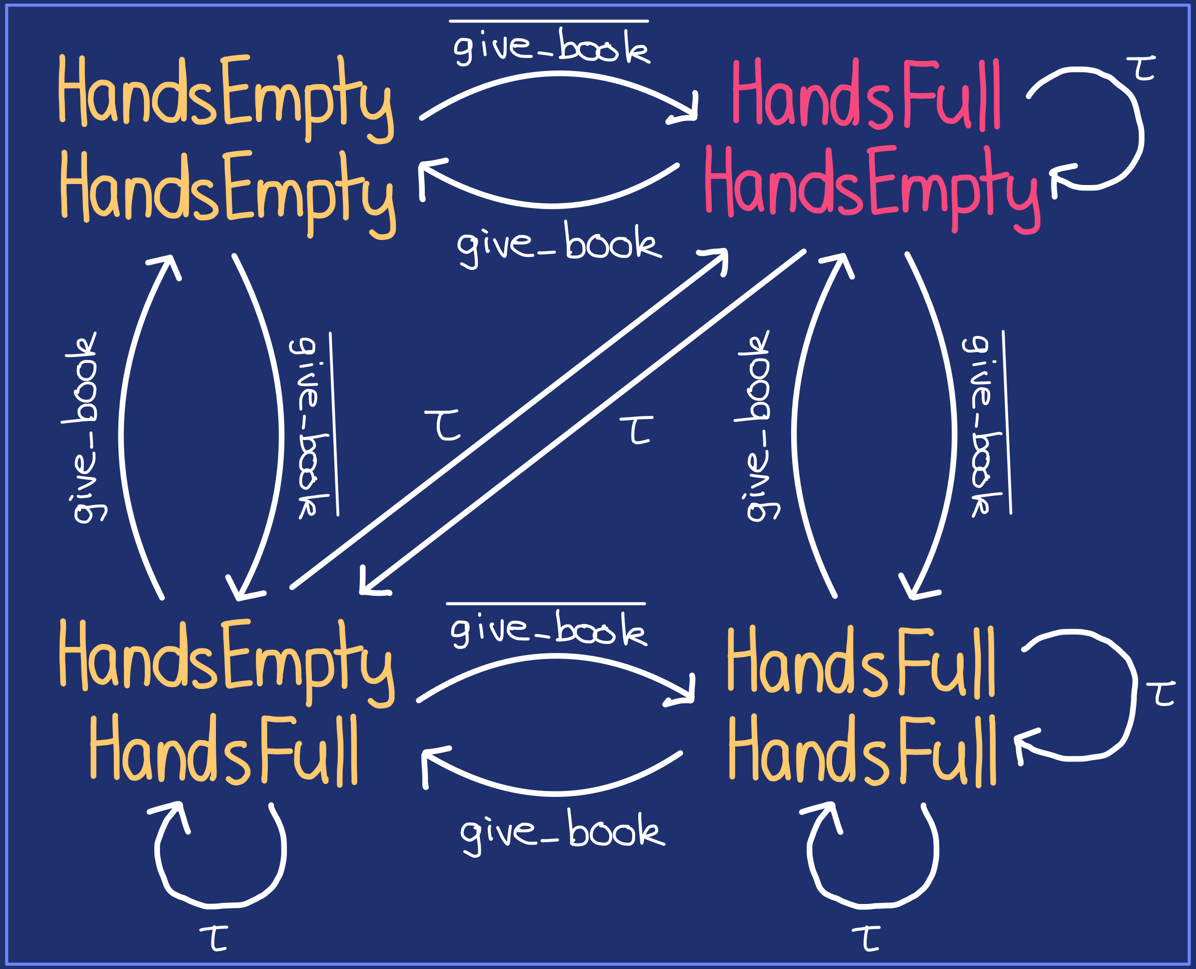
Reader1



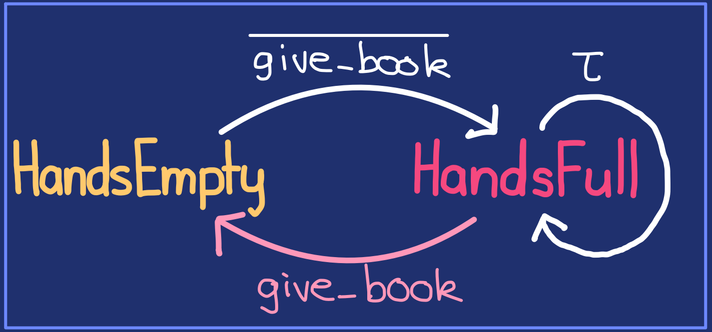


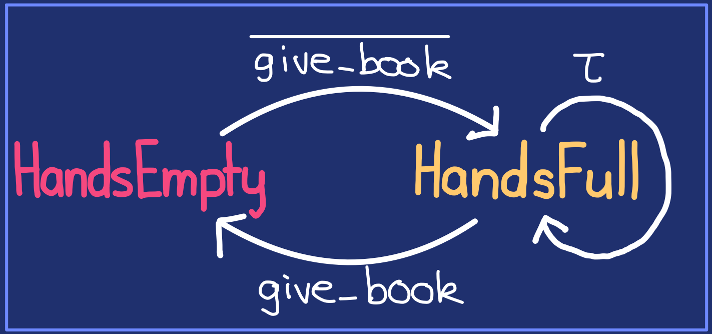
Reader2

ReaderPair



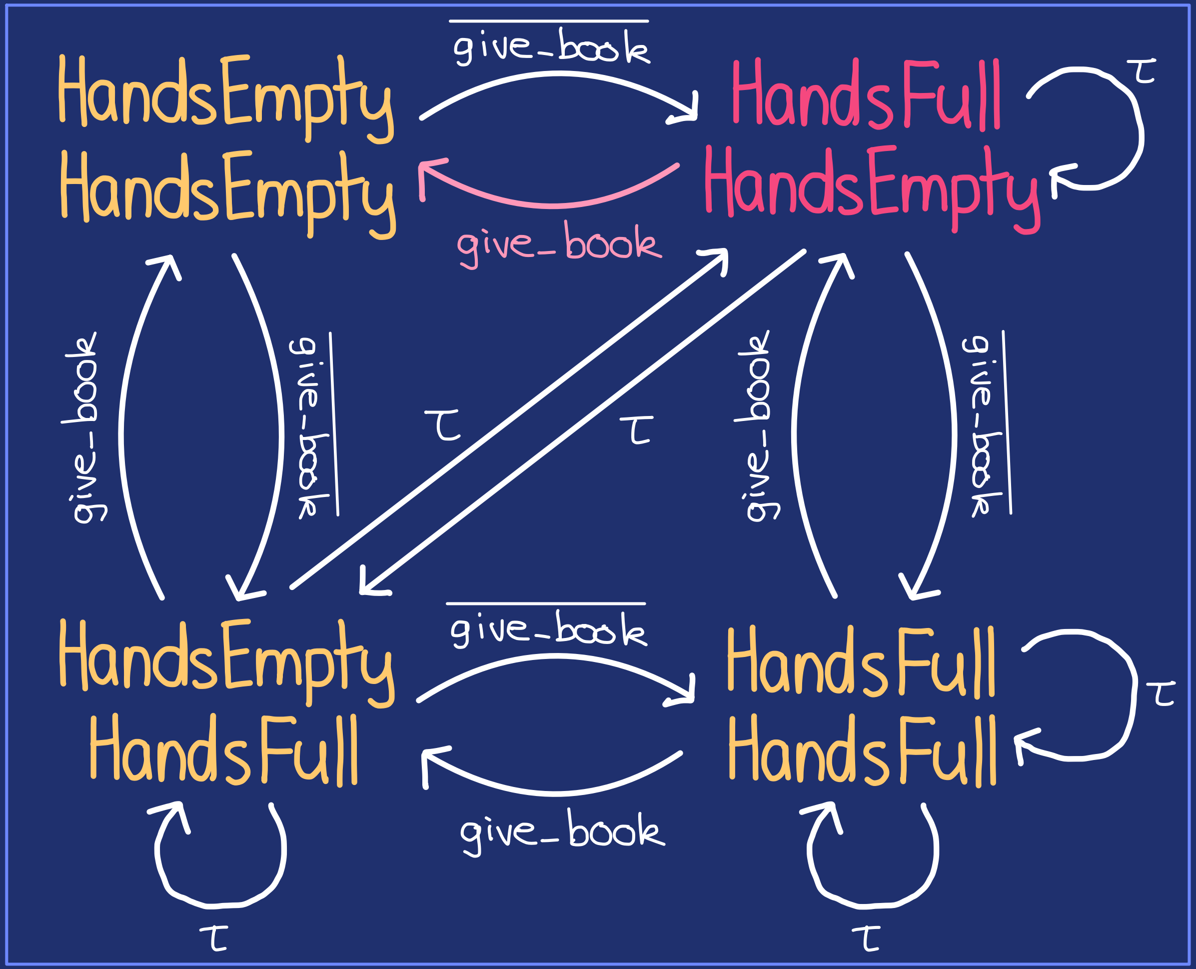
Reader1





Reader2

ReaderPair



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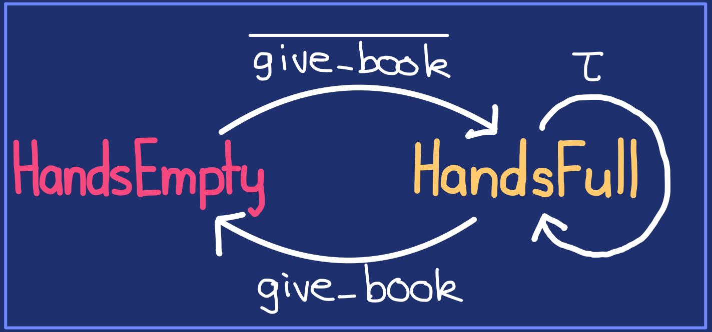
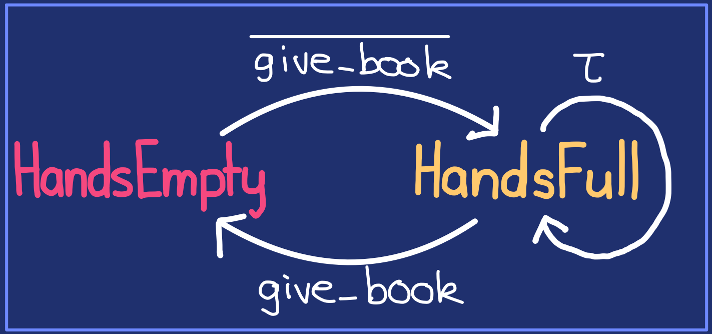
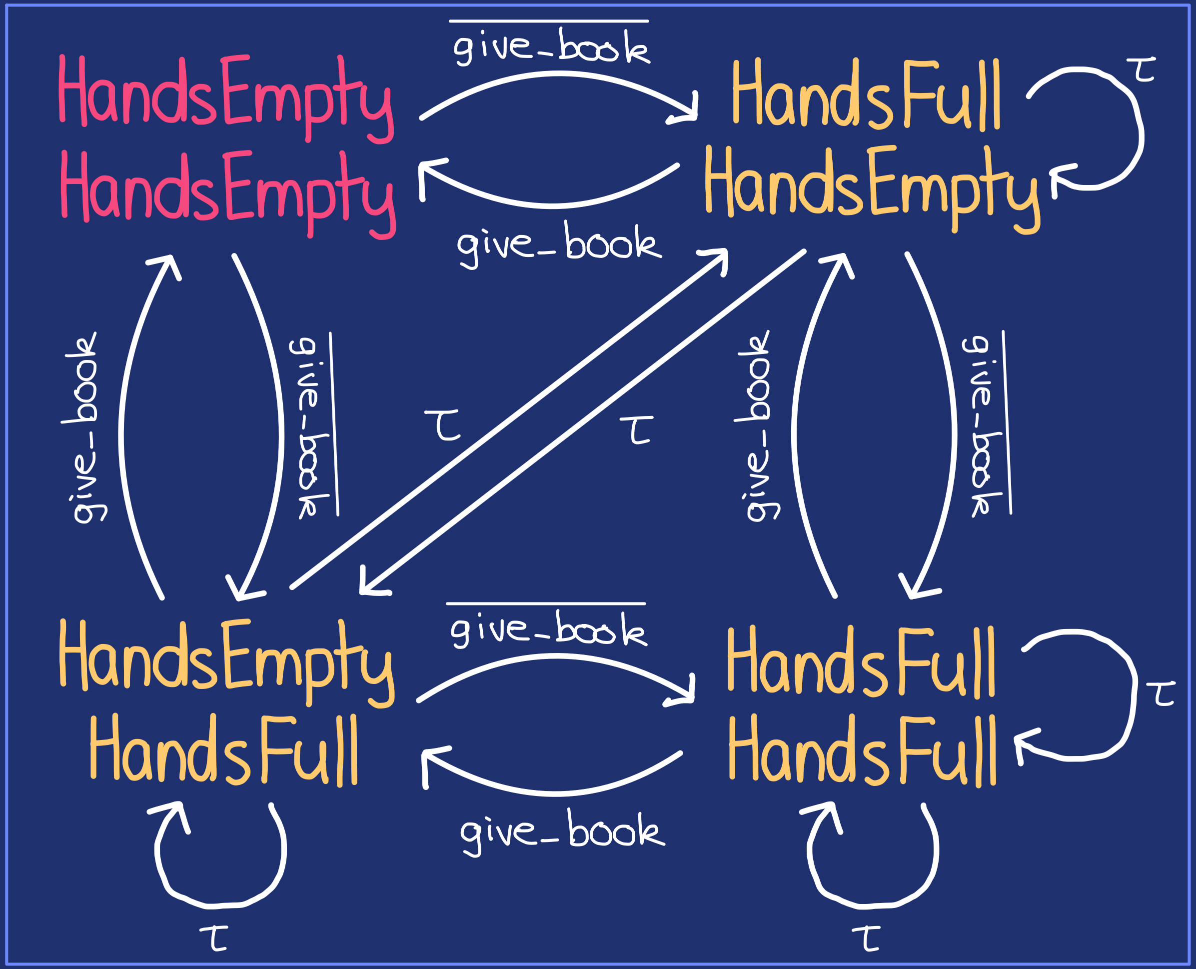


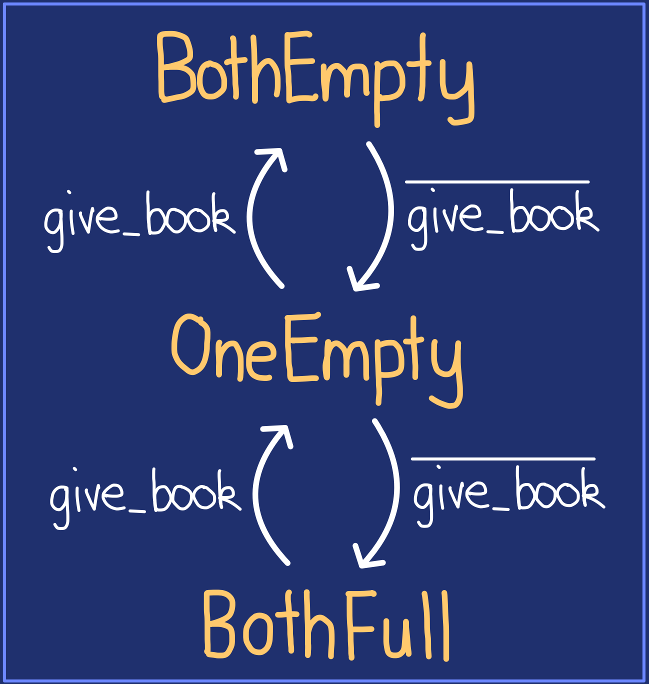


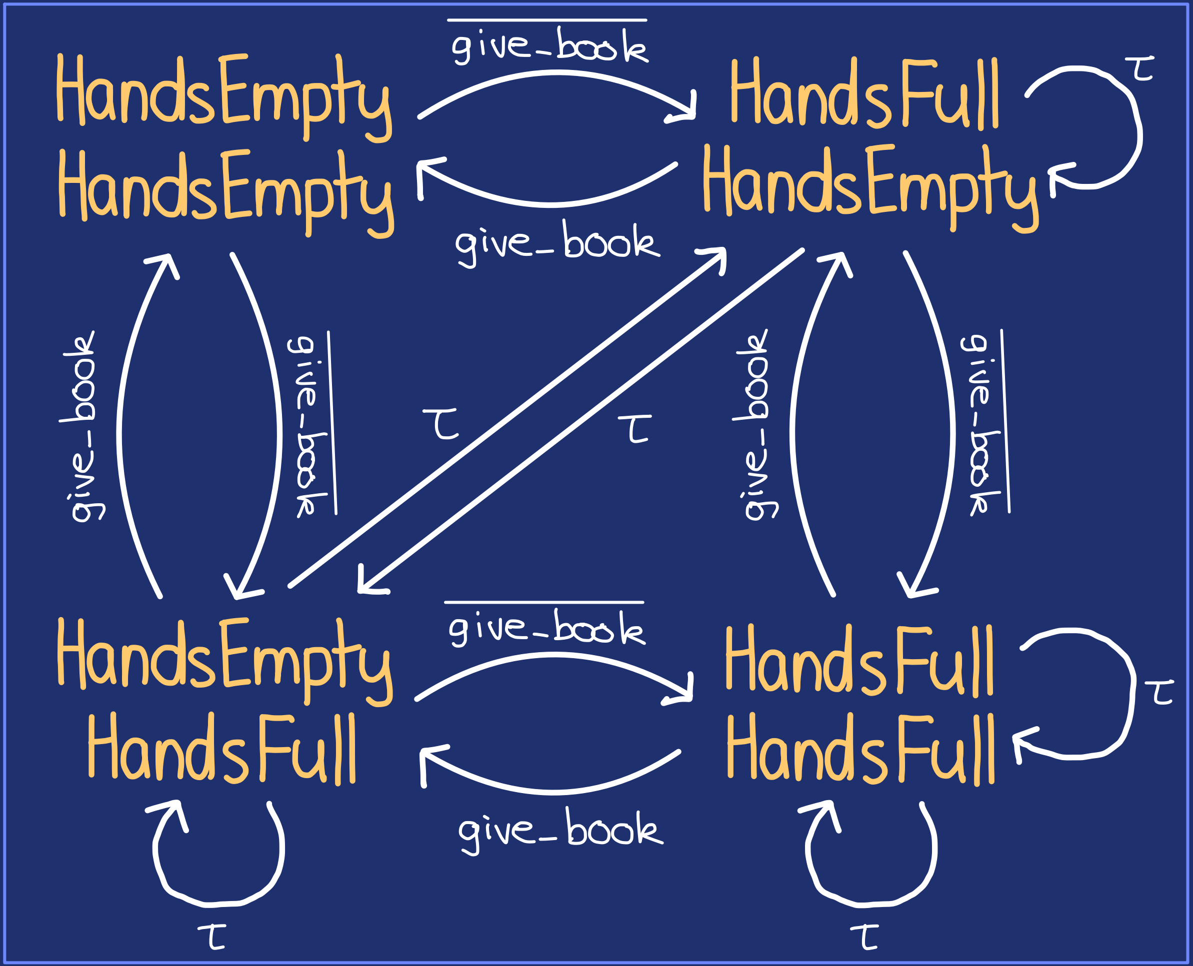
Reader2

ReaderPair



Two different systems can have the same observable behaviour





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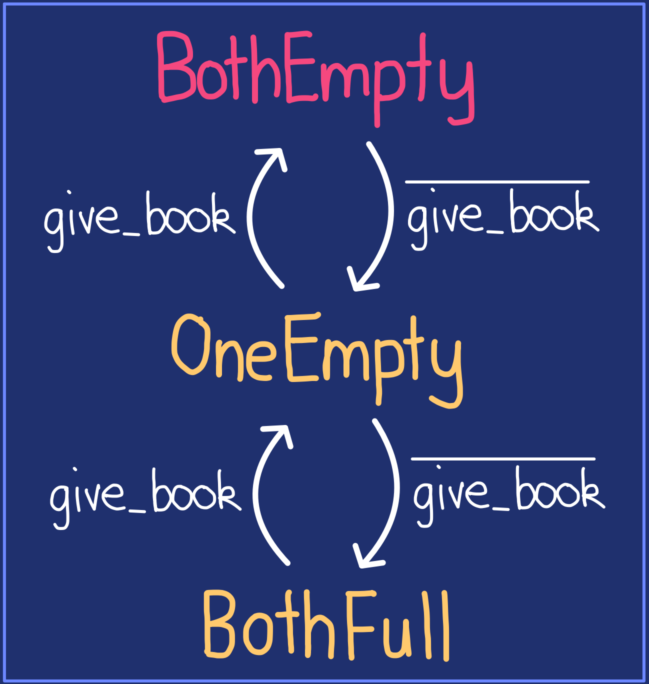

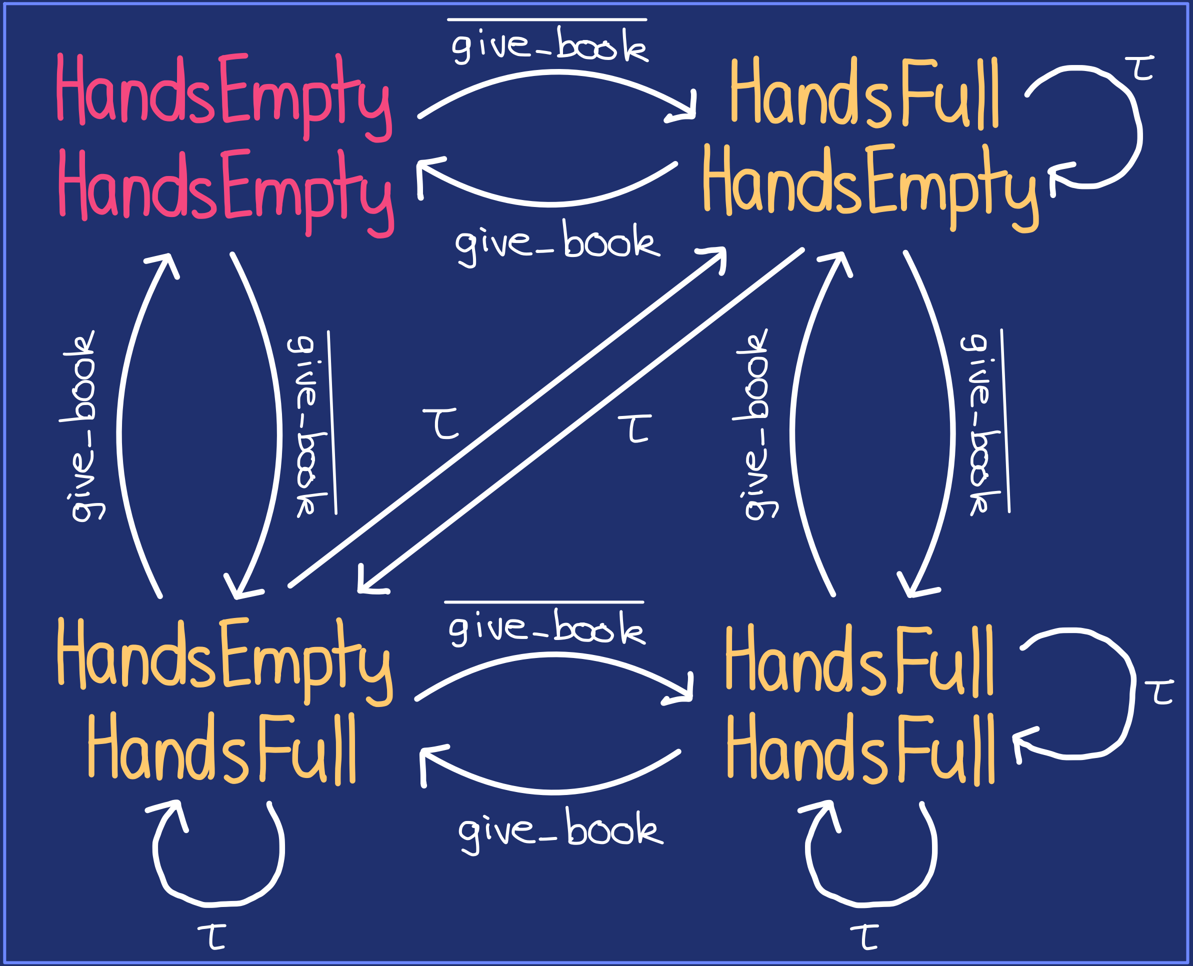

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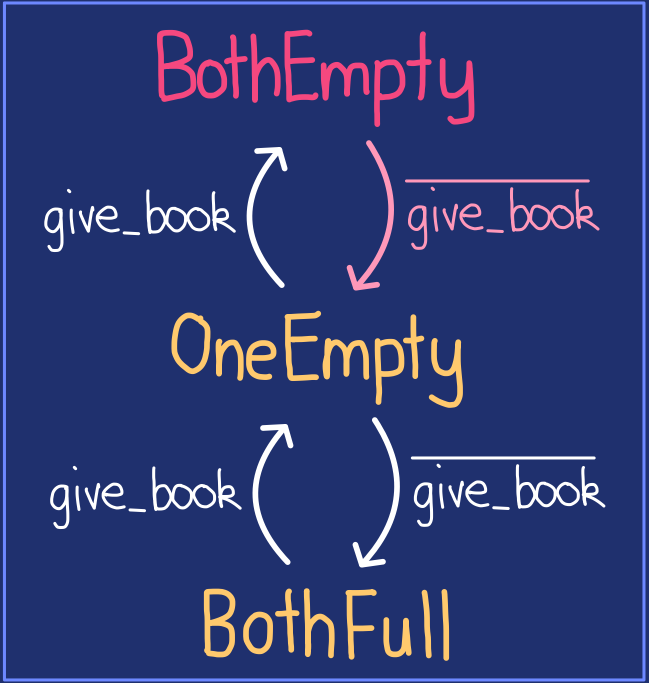

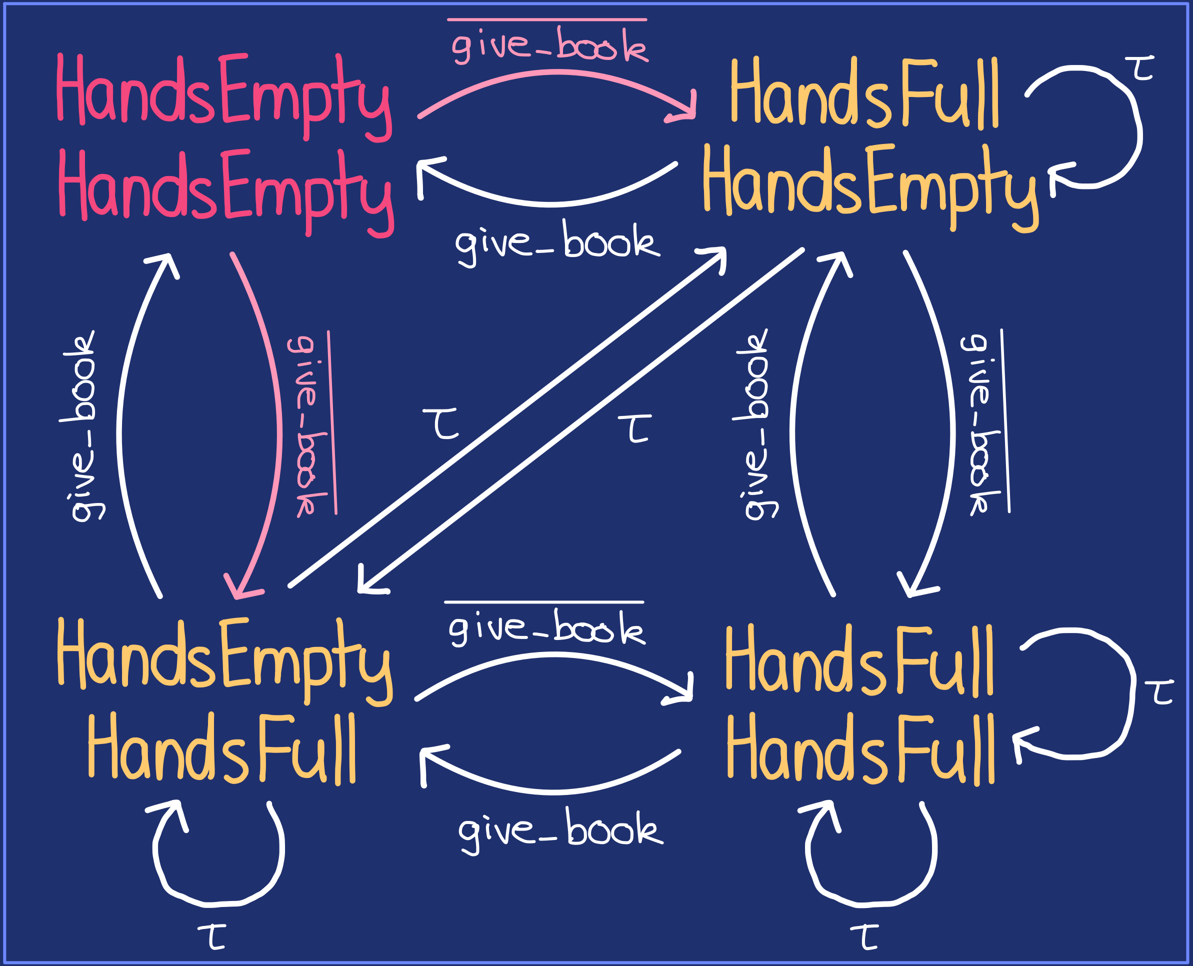

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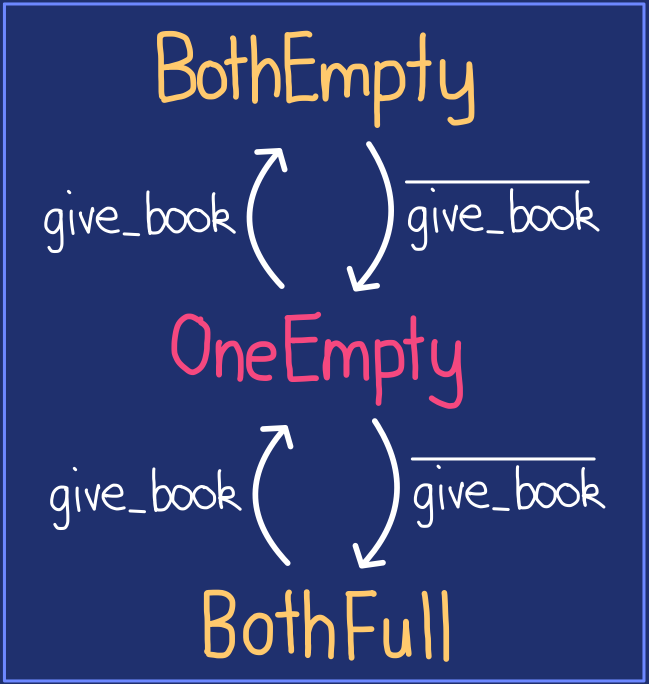

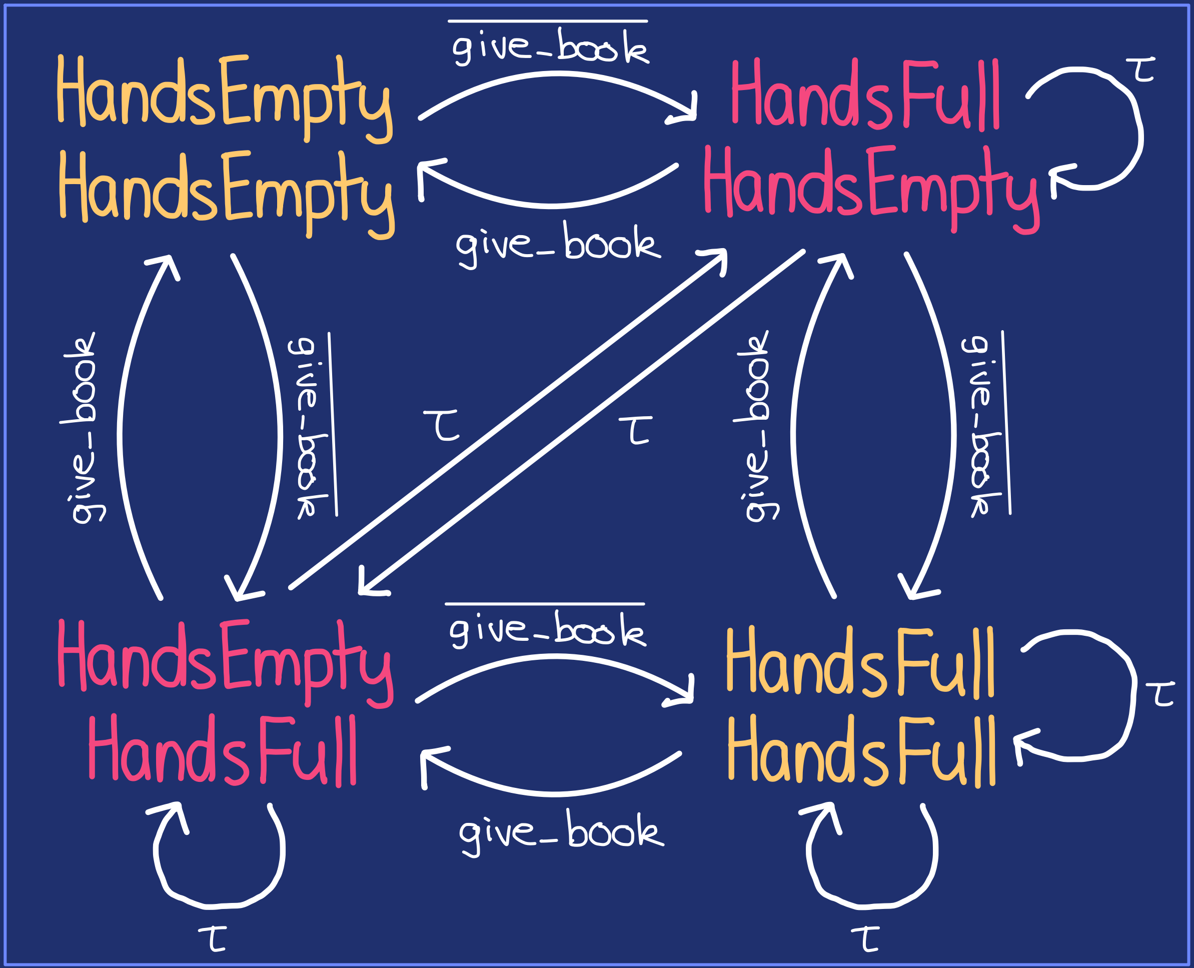

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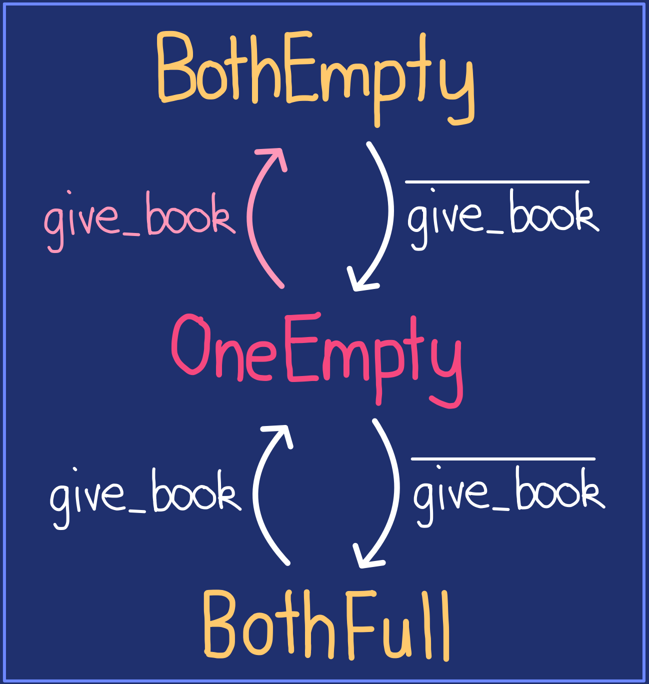

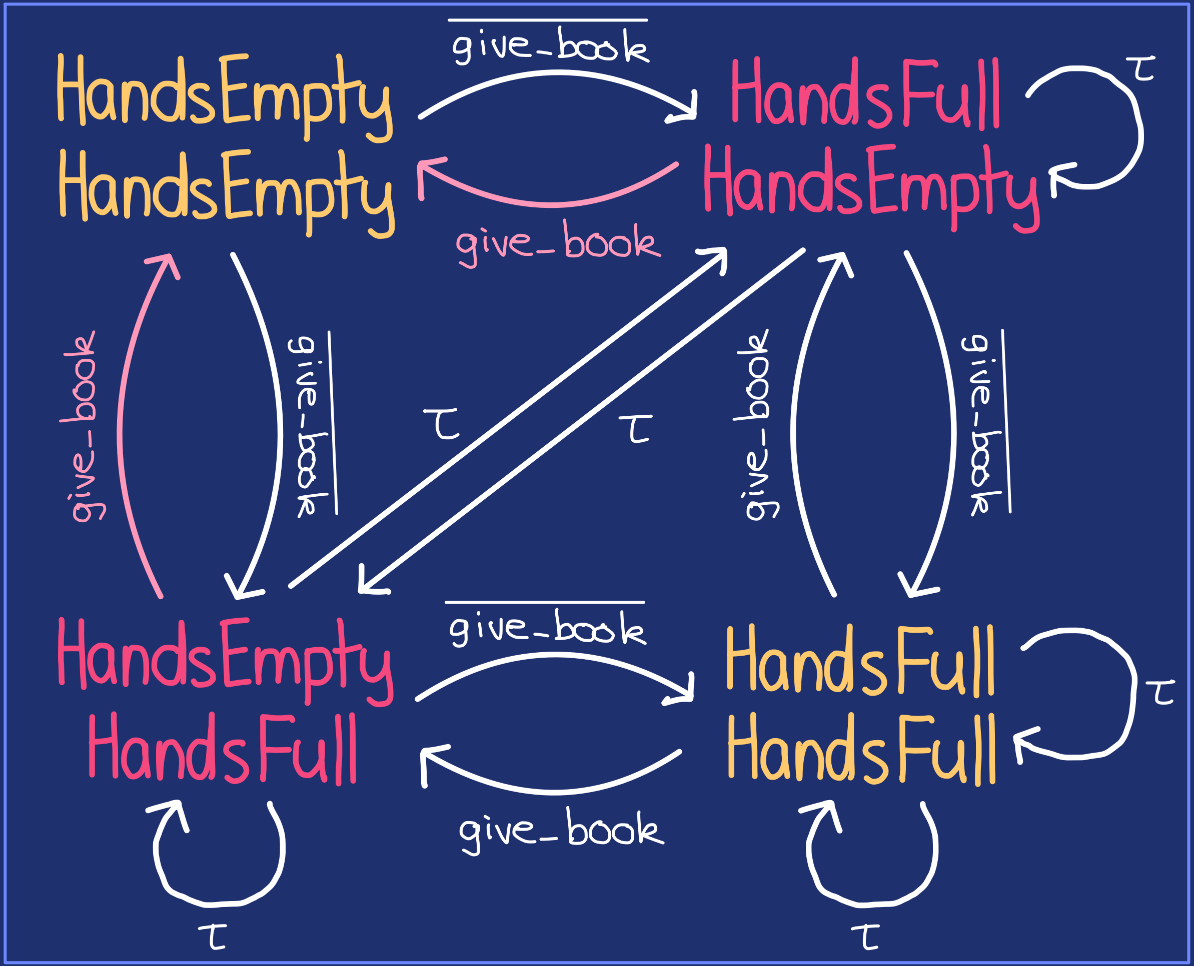

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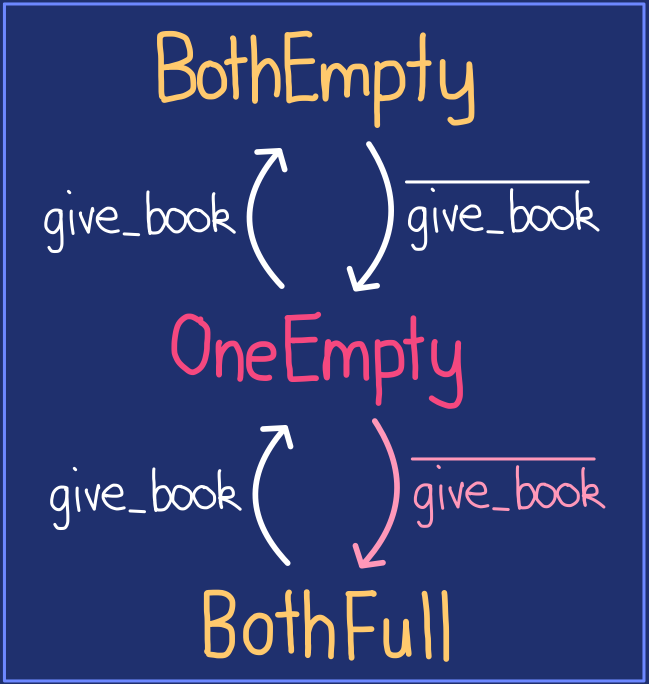

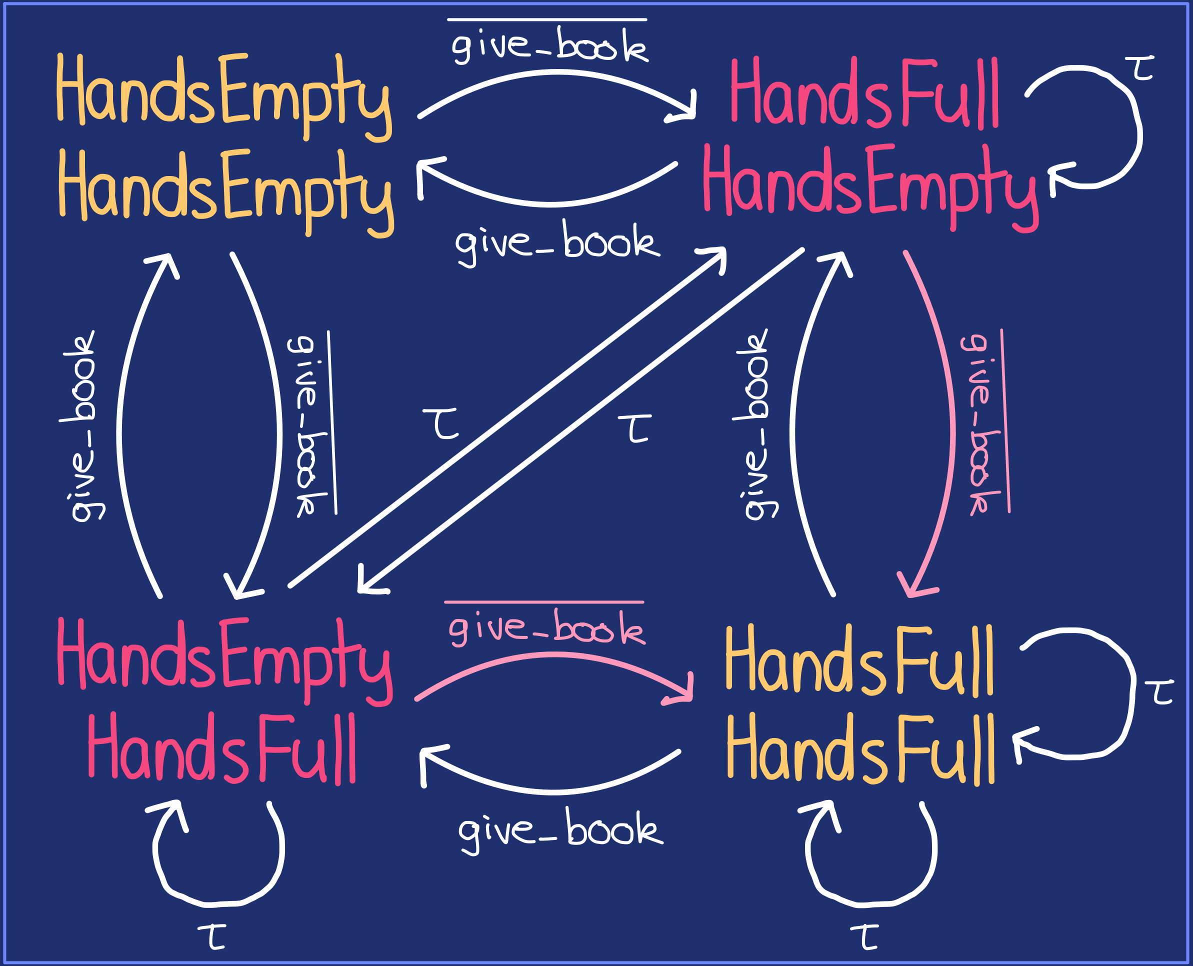

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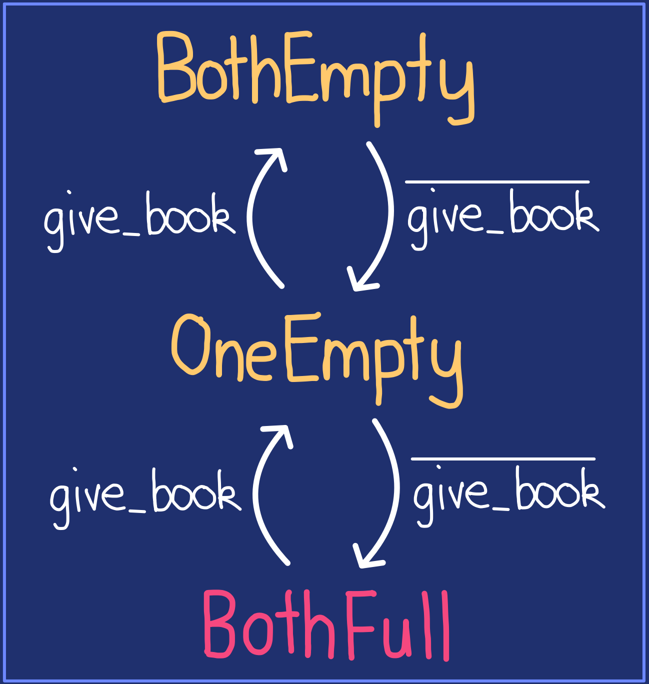

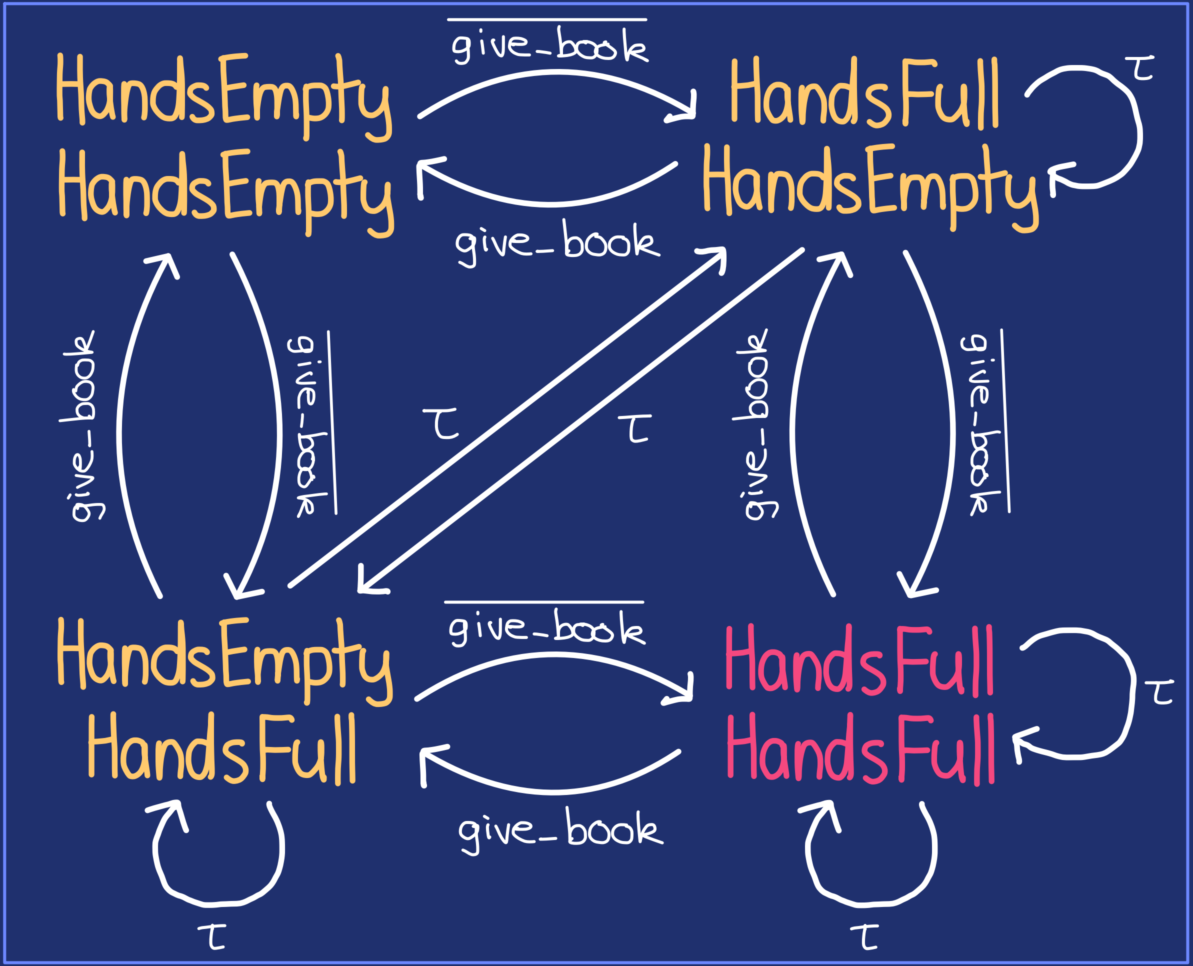

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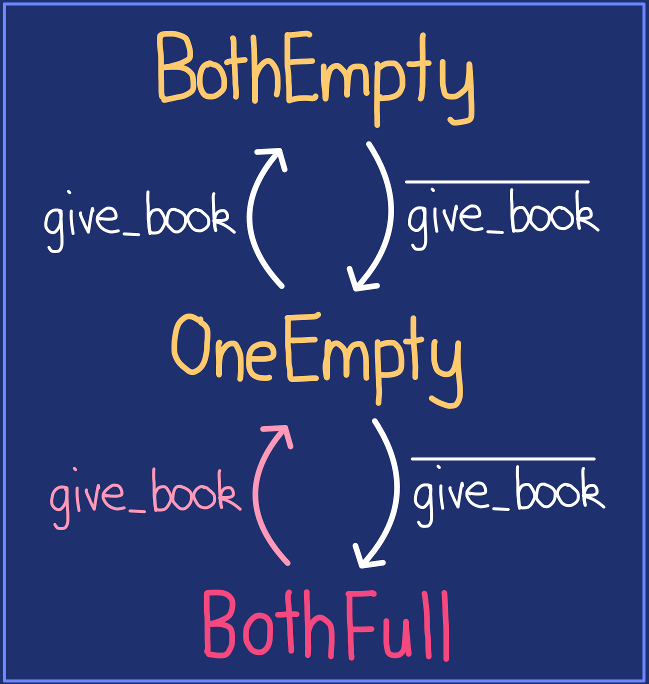
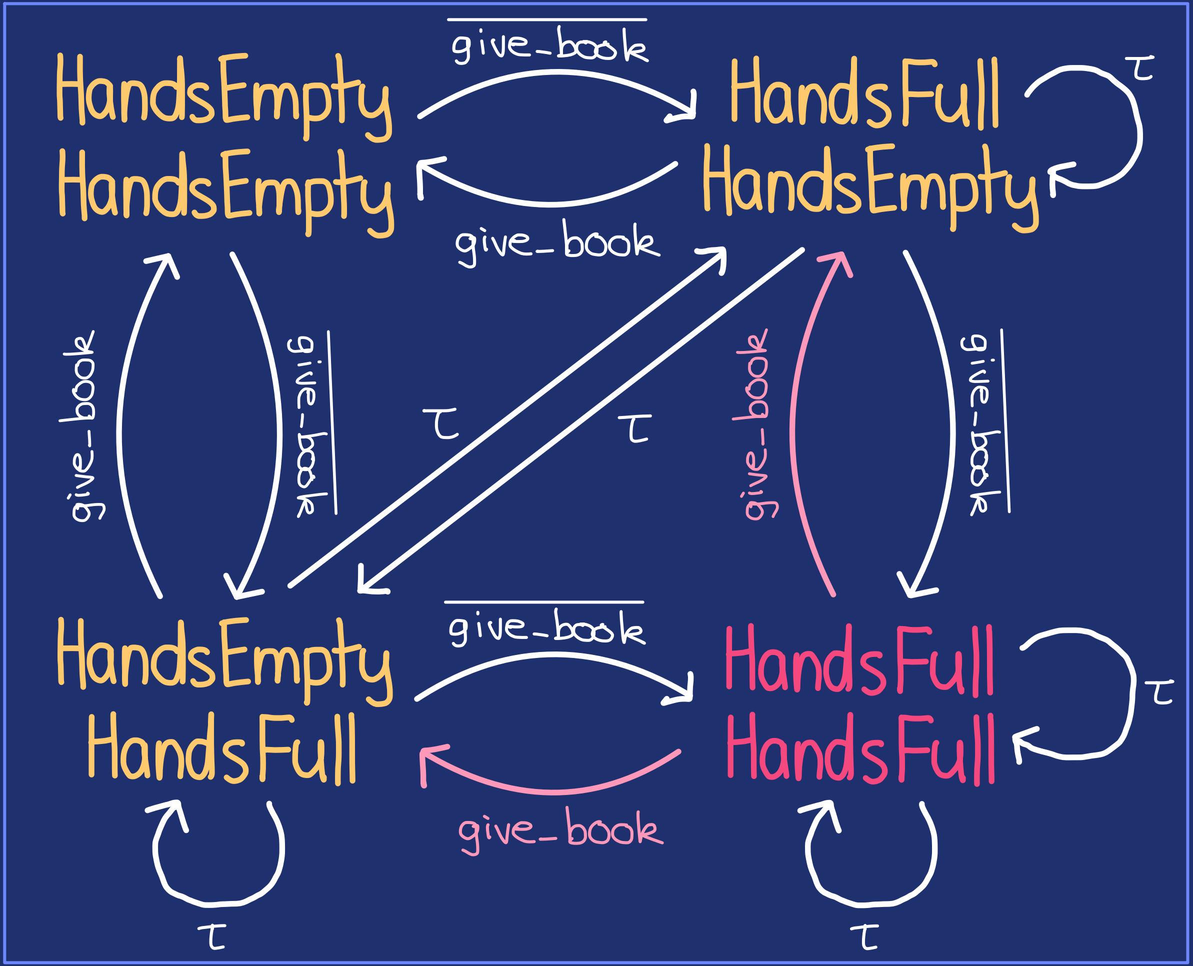
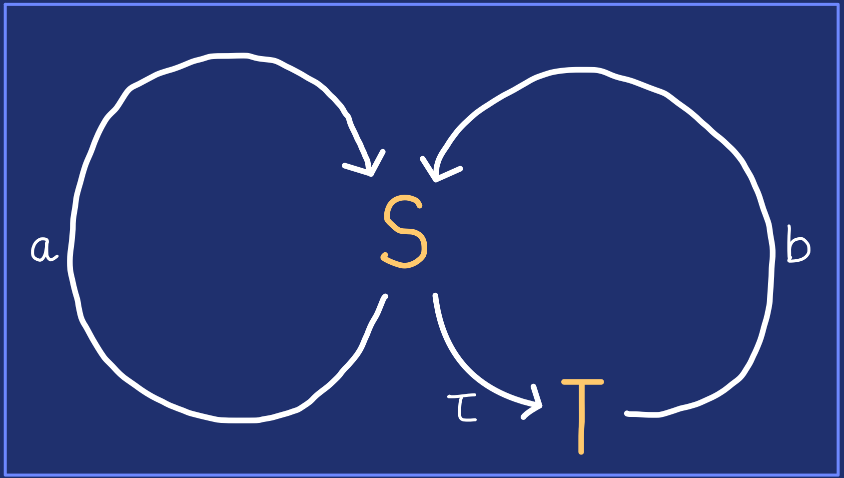
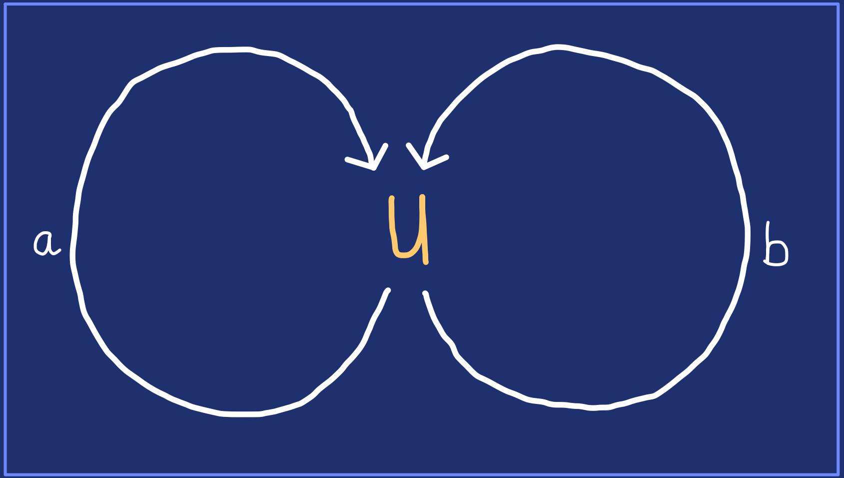




Internal actions can sometimes be observed indirectly





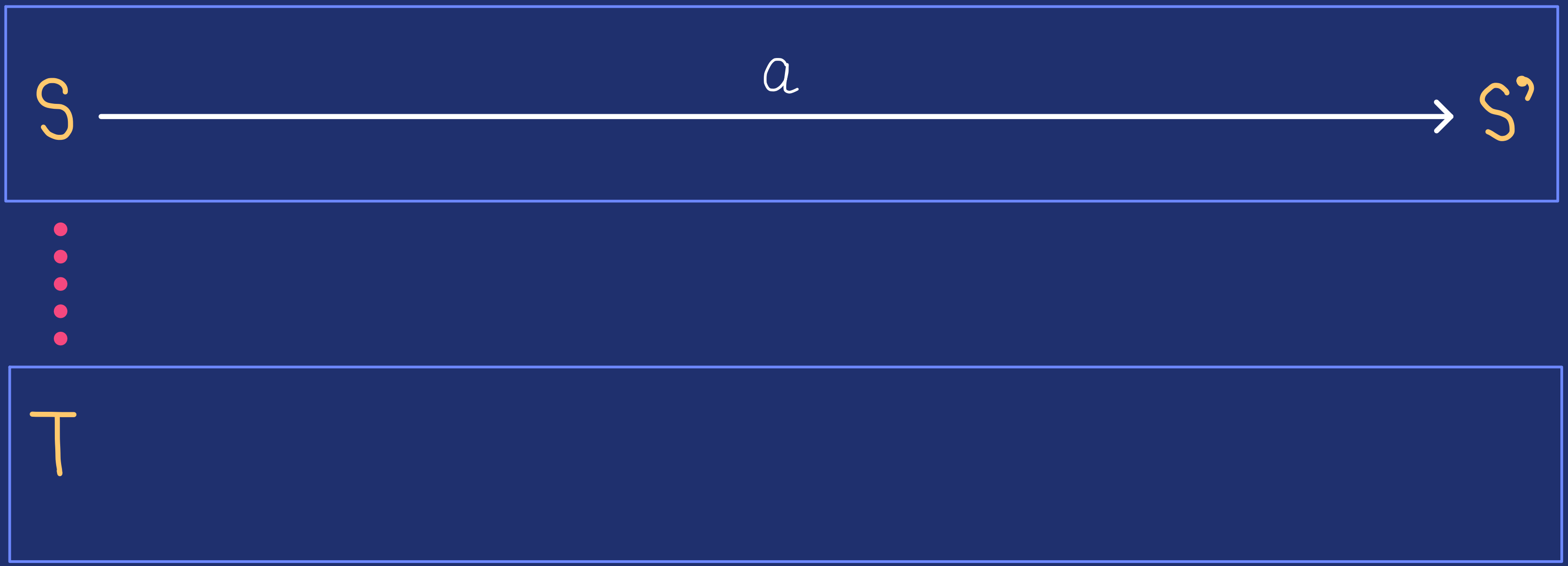
These systems are not observationally equivalent

Given a set $\mathcal{A}$ of actions (and their complements)

systems are modelled as
$\mathcal{A} \sqcup \{\tau\}$-labelled transition systems

behavioural equivalence is captured by
weak bisimulation

A weak bisimulation between two systems is a relation between their sets of states such that



$a \neq \tau$

A weak bisimulation between two systems is a relation between their sets of states such that

$$S \xrightarrow{a} S'$$

$$T \xrightarrow[\tau]{} \cdots \xrightarrow[\tau]{} U \xrightarrow[a]{} U' \xrightarrow[\tau]{} \cdots \xrightarrow[\tau]{} T'$$

$$a \neq \tau$$

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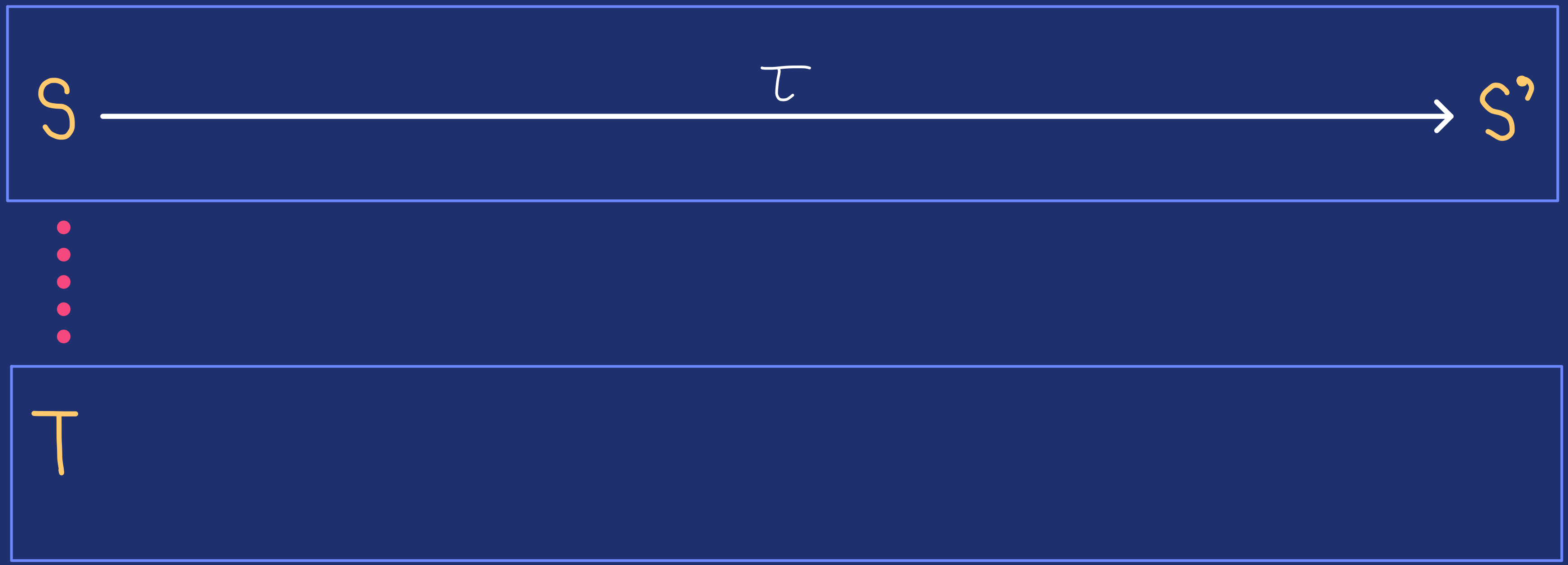
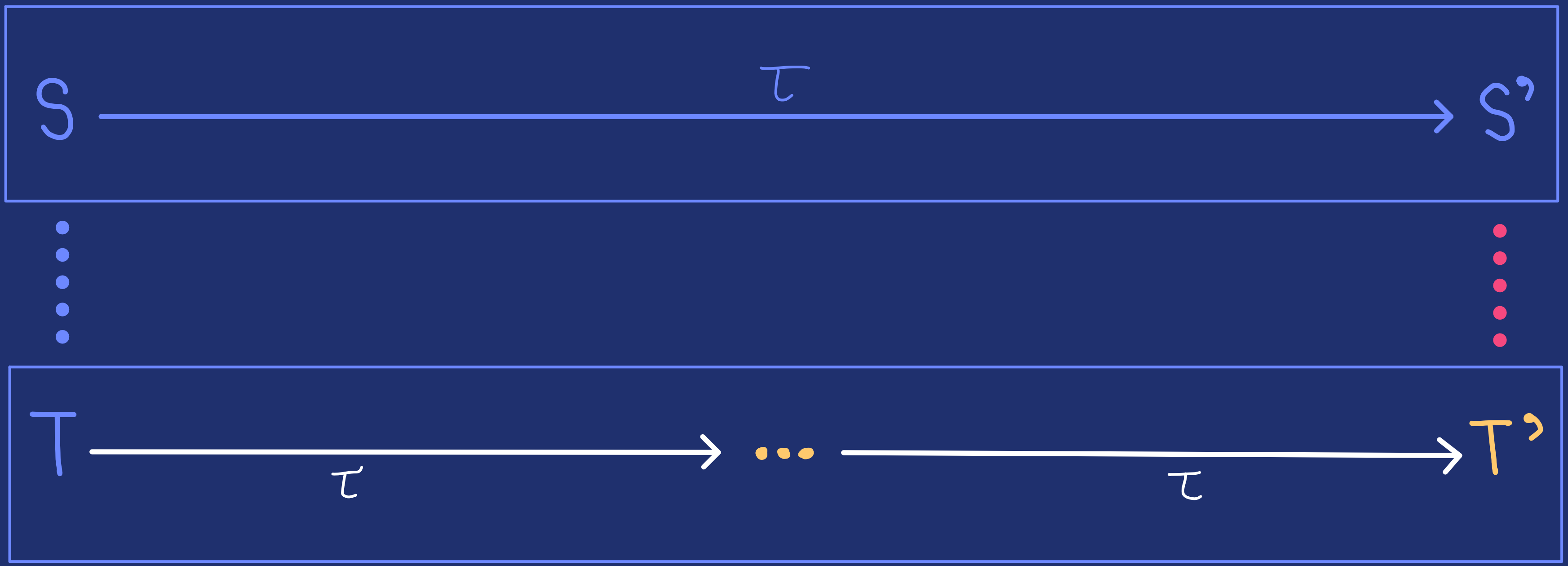


A *weak bisimulation* between two systems is a relation between their sets of states such that



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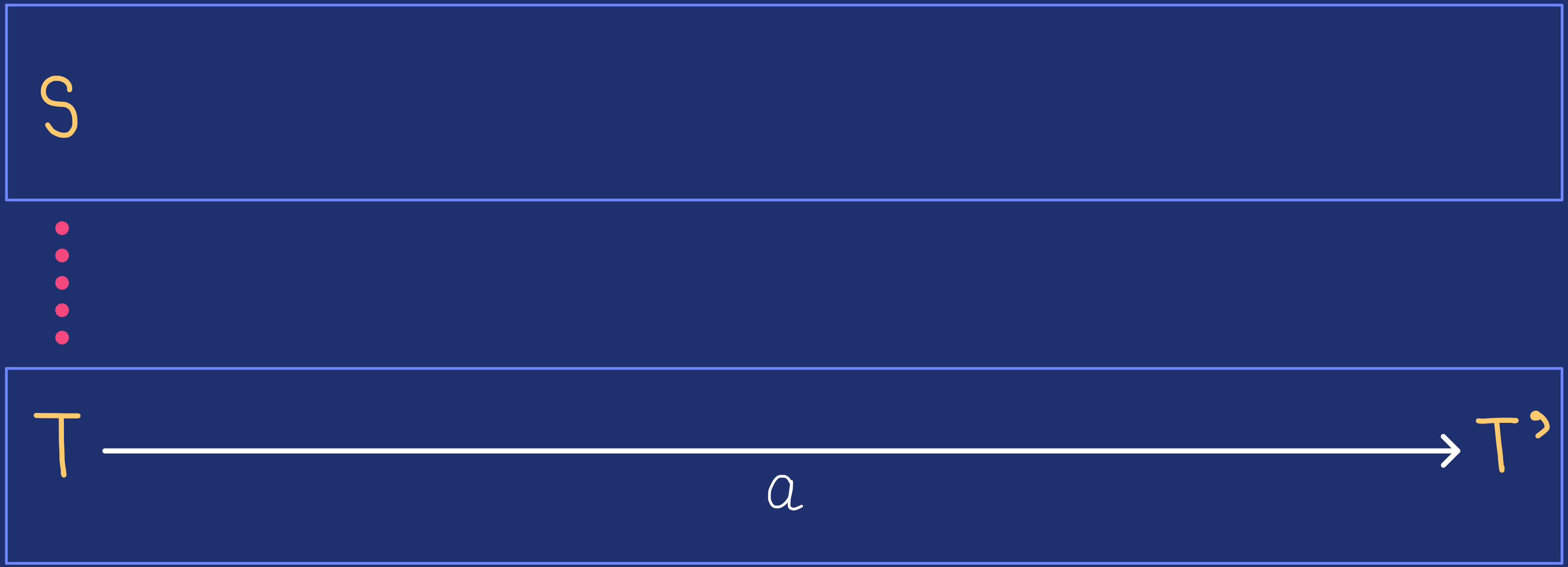


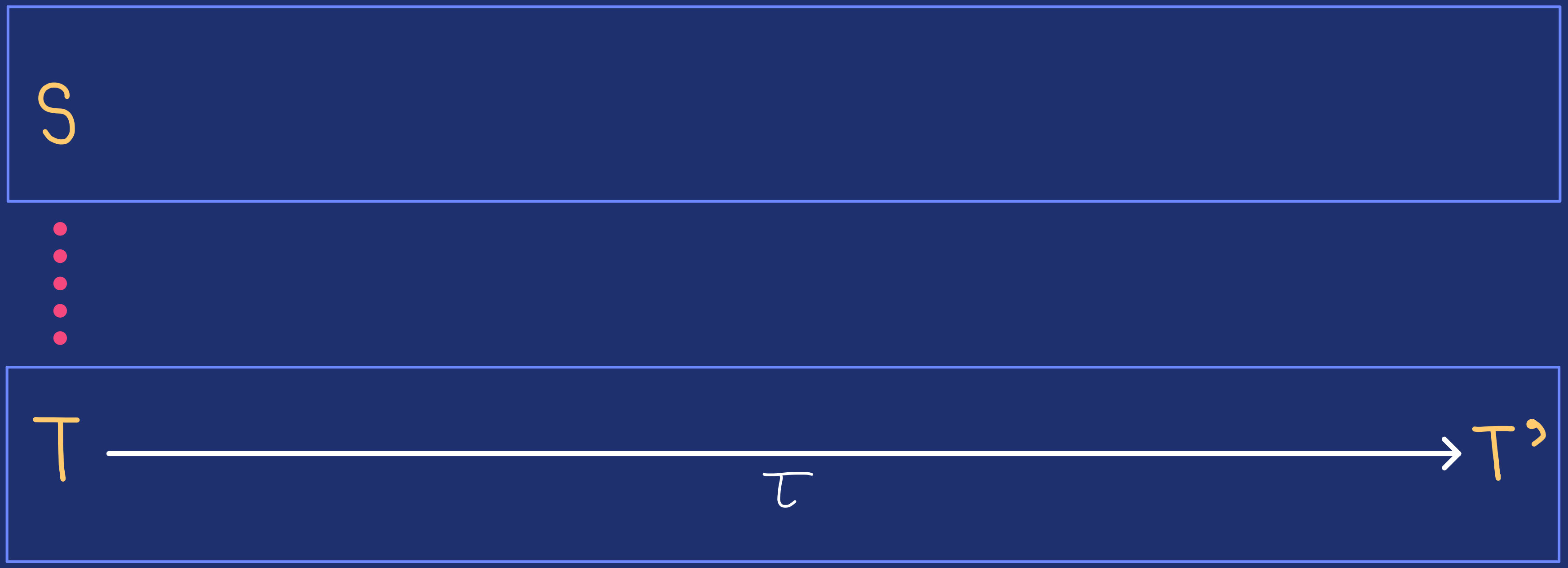
$a \neq \tau$

A *weak bisimulation* between two systems is a relation between their sets of states such that

$$\begin{tikzcd}
S \arrow[r, "\tau"] \arrow[d, dotted, no head] & \cdots \arrow[r, "\tau"] & u \arrow[r, "a"] & u' \arrow[r, "\tau"] & \cdots \arrow[r, "\tau"] & S' \arrow[d, dotted, no head] \\
T \arrow[rrrrr, "a"'] & & & & & T'
\end{tikzcd}$$

$$a \neq \tau$$

A weak bisimulation between two systems is a relation between their sets of states such that



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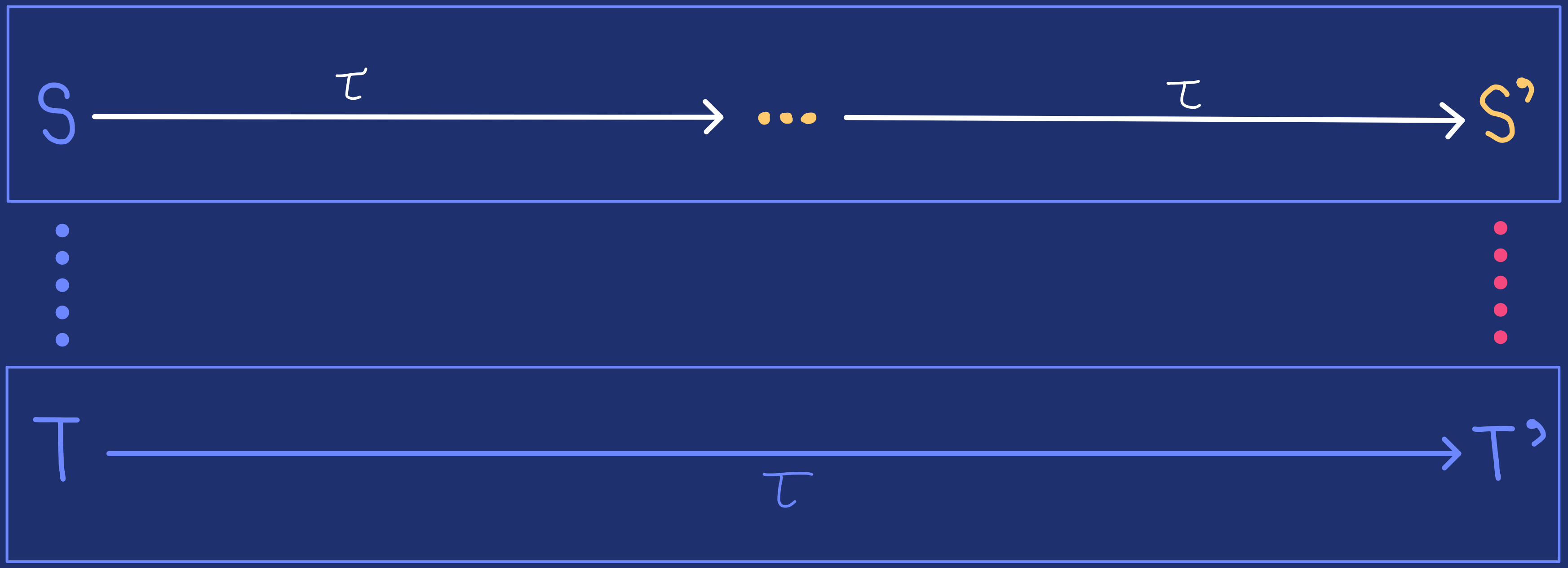

Enriched Graphs and Categories

Enriched Graphs

A suplattice is a partially ordered set $(L, \leqslant)$ with suprema.

Examples:

- $\mathbb{B} = \{\bot, \top\}$ ordered by $\Rightarrow$
- $P(A)$ ordered by $\subseteq$
- $[0, \infty]$ ordered by $\geqslant$

An L-enriched graph $\mathcal{G}$ is

- a set $ob(\mathcal{G})$ of objects
- $\mathcal{G}(a,b) \in L$ for all $a, b \in ob(\mathcal{G})$

Examples:

- Kripke frames
- A-labelled transition systems
- weighted graphs

Enriched Categories

A *quantale* is a suplattice $(Q, \leqslant)$ and a monoid $(Q, \otimes, I)$ such that $X \otimes \_$ and $\_ \otimes X$ preserve suprema.

Examples:

- $\mathbb{B} = \{\bot, \top\}$ with $\wedge$ and $\top$
- $P(A^*)$ with $S \otimes T = \{st : s \in S, t \in T\}$, $I = \{\varepsilon\}$
- $[0, \infty]$ with $+$ and 0

A *Q-enriched category* $\mathcal{C}$ is a Q-enriched graph such that $I \leqslant \mathcal{C}(a,a)$ and $\mathcal{C}(a,b) \otimes \mathcal{C}(b,c) \leqslant \mathcal{C}(a,c)$

Examples:

- reflexive transitive Kripke frames
- generalised A-labelled transition systems
- Lawvere metric spaces

Enriched Lenses

Let $(L, \leqslant)$ be a suplattice, and let $\mathcal{G}$ and $\mathcal{H}$ be L-enriched graphs.

A lens $F : \mathcal{G} \to \mathcal{H}$ is a function $F : \mathrm{ob}(\mathcal{G}) \to \mathrm{ob}(\mathcal{H})$ such that

$$\mathcal{G}(a,b) \leqslant \mathcal{H}(Fa, Fb)$$

and

$$\mathcal{H}(Fa, y) \leqslant \sup_{b \in F^{-1}\{y\}} \mathcal{G}(a,b).$$

Enriched Lenses

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and

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Example (Kripke frames):

$$(L,\leqslant) = (\mathbb{B},\Rightarrow)$$

p-morphism or zig-zag morphism

$$a \longrightarrow b \;\Rightarrow\; Fa \longrightarrow Fb$$

$$Fa \longrightarrow y \;\Rightarrow\; \exists b \in F^{-1}\{y\}.\; a \longrightarrow b$$

Enriched Lenses

Let $(Q, \leqslant, \otimes, I)$ be a quantale, and let $\mathcal{C}$ and $\mathcal{D}$ be L-enriched categories.

A lens $F : \mathcal{C} \to \mathcal{D}$ is a function $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ such that

$$\mathcal{C}(a,b) \leqslant \mathcal{D}(Fa, Fb)$$

and

$$\mathcal{C}(Fa, y) \leqslant \sup_{b \in F^{-1}\{y\}} \mathcal{D}(a,b).$$

Example (labelled transition systems):

$$(Q, \leqslant, \otimes, I) = (P(A^*), \subseteq, \otimes, \{\varepsilon\})$$

functional weak bisimulation

$$a \xrightarrow{s} b \implies Fa \xrightarrow{s} Fb$$

$$Fa \xrightarrow{s} y \implies \exists b \in F^{-1}\{y\}.\ a \xrightarrow{s} b$$

ENRICHED LENSES

Let $(\mathcal{Q},\leq,\otimes,I)$ be a quantale, and let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{L}$-enriched categories.

A lens $F:\mathcal{C}\to\mathcal{D}$ is a function $F:ob(\mathcal{C})\to ob(\mathcal{D})$ such that

$$\mathcal{C}(a,b)\leq\mathcal{D}(Fa,Fb)$$

and

$$\mathcal{C}(Fa,y)\leq\sup_{b\in F^{-1}\{y\}}\mathcal{D}(a,b).$$

Example (lawvere metric spaces):

$$(\mathcal{Q},\leq,\otimes,I)=([0,\infty],\geq,+,0)$$

weak submetry

$$d(a,b)\geq d(Fa,Fb)$$

and

$$d(Fa,y)\geq\inf_{b\in F^{-1}\{y\}}d(a,b).$$

Enriched Bisimulations

Let $(L, \leq)$ be a suplattice, and let $\mathcal{G}$ and $\mathcal{H}$ be L-enriched graphs.

A bisimulation $R : \mathcal{G} \nrightarrow \mathcal{H}$ is a relation $R : ob(\mathcal{G}) \nrightarrow ob(\mathcal{H})$ such that if $a \, R \, x$ then

$$\mathcal{G}(a, b) \leq \sup_{y : bRy} \mathcal{H}(x, y)$$

and

$$\mathcal{H}(x, y) \leq \sup_{b : bRy} \mathcal{G}(a, b).$$

Enriched Bisimulations

Let $(L,\leqslant)$ be a suplattice, and let $\mathcal{G}$ and $\mathcal{H}$ be L-enriched graphs.

A bisimulation $R:\mathcal{G} \nrightarrow \mathcal{H}$ is a relation $R:ob(\mathcal{G}) \nrightarrow ob(\mathcal{H})$ such that if $a\,R\,x$ then

$$\mathcal{G}(a,b) \leqslant \sup_{y:bRy} \mathcal{H}(x,y)$$

and

$$\mathcal{H}(x,y) \leqslant \sup_{b:bRy} \mathcal{G}(a,b).$$

Example (Kripke frames):

$$(L,\leqslant) = (\mathbb{B},\Rightarrow)$$

p-relation or zig-zag relation

$$a \to b \implies \exists y \in R\{b\}.\ x \to y$$

$$x \to y \implies \exists b \in R^{-1}\{y\}.\ a \to b$$

Enriched Bisimulations

Let $(Q,\leq,\otimes,I)$ be a quantale, and let $\mathcal{C}$ and $\mathcal{D}$ be L-enriched categories.

A **bisimulation** $R : \mathcal{C} \nrightarrow \mathcal{D}$ is a relation $R : ob(\mathcal{C}) \nrightarrow ob(\mathcal{D})$ such that if $a\,R\,x$ then

$$\mathcal{G}(a,b) \leq \sup_{y : bRy} \mathcal{H}(x,y)$$

and

$$\mathcal{H}(x,y) \leq \sup_{b : bRy} \mathcal{G}(a,b).$$

Example (labelled transition systems):

$$(Q,\leq,\otimes,I) = (P(A^*), \subseteq, \otimes, \{\varepsilon\})$$

weak bisimulation

$$a \xrightarrow{s} b \implies \exists y \in R\{b\}.\ x \xrightarrow{s} y$$

$$x \xrightarrow{s} y \implies \exists b \in R^{-1}\{y\}.\ a \xrightarrow{s} b$$

Proposition:

$$A \xleftarrow{f_1} \mathcal{X} \xrightarrow{f_2} B \quad \mapsto \quad A \xrightarrow[\quad]{\mathrm{Im}\langle f_1, f_2\rangle} B$$

spans of $\mathcal{V}$-enriched lenses $\xrightarrow{\text{split surjection}}$ $\mathcal{V}$-enriched bisimulations

$$A \xleftarrow[\pi_1]{} R \xrightarrow[\pi_2]{} B \quad \mapsfrom \quad A \xrightarrow[R]{} B$$

$$\mathrm{ob}(R) = R \subseteq \mathrm{ob}(A) \times \mathrm{ob}(B)$$

$$R\left(\binom{a}{b}, \binom{a'}{b'}\right) = A(a, a') \wedge B(b, b')$$

Summary

Enriched bisimulations

- generalise several common kinds of bisimulation
- are equivalence classes of spans of enriched lenses

Questions

- What are the bisimulations for other common quantales?
- What other parts of bisimulation theory generalise?

https://mdimeglio.github.io
https://bryceclarke.github.io