Universal indexed categories

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Categorical Late Lunch

Outline

Motivation for generalised indexed families

- A $\mathcal{V}$-enriched category $\mathbf{C}$ has a set $\mathbf{C}_0$ and a $(\mathbf{C}_0 \times \mathbf{C}_0)$-indexed family of objects of $\mathcal{V}$
$$\big(\mathbf{C}(X, Y)\big)_{(X,Y) \in \mathbf{C}_0 \times \mathbf{C}_0}.$$
- An $\mathbf{E}$-internal category $\mathbf{C}$ has an object $\mathbf{C}_0$ of $\mathbf{E}$ and a morphism of $\mathbf{E}$
$$\langle s, t \rangle \colon \mathbf{C}_1 \to \mathbf{C}_0 \times \mathbf{C}_0.$$
- A $\mathcal{V}$-enriched functor $F \colon \mathbf{C} \to \mathbf{D}$ consists of a function $F_0 \colon \mathbf{C}_0 \to \mathbf{D}_0$ and a $(\mathbf{C}_0 \times \mathbf{C}_0)$-indexed family of morphisms of $\mathcal{V}$
$$\big(\mathbf{C}(X, Y) \xrightarrow{F_{X,Y}} \mathbf{D}(F_0 X, F_0 Y)\big)_{(X,Y) \in \mathbf{C}_0 \times \mathbf{C}_0}.$$
- An $\mathbf{E}$-internal functor $F \colon \mathbf{C} \to \mathbf{D}$ consists of a morphism $F_0 \colon \mathbf{C}_0 \to \mathbf{D}_0$ of $\mathbf{E}$ and a morphism of $\mathbf{E}$
$$F_1 \colon \mathbf{C}_1 \to (\mathbf{C}_0 \times \mathbf{C}_0) \times_{(\mathbf{D}_0 \times \mathbf{D}_0)} \mathbf{D}_1.$$

What does it mean to have a family of objects or morphisms indexed by something other than a set?

Let $\mathbf{S}$ be a category.

Definition

An ***S-indexed category*** $\mathbb{C}$ is a pseudofunctor $\mathbb{C}\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Cat}$.

Write $\mathbf{C}^J$ instead of $\mathbb{C}(J)$ and Δ_r instead of $\mathbb{C}(r)$.

Suppose that $\mathbf{S}$ has a chosen terminal object 1.

Definition

The ***underlying category*** of an $\mathbf{S}$-indexed category $\mathbb{C}$ is the category $\mathbf{C}^1$.
We also call $\mathbb{C}$ an ***indexing*** of $\mathbf{C}^1$.

Write Δ_J instead of $\Delta_{!_J}$ where $!_J\colon J \to 1$.

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Think of Δ_J as a diagonal functor and Δ_r as an indexed diagonal functor.

Definition

An indexed category has ***indexed sums*** if each Δ_r has a left adjoint Σ_r and these satisfy the left Beck-Chevalley condition.

Write Σ_J instead of $\Sigma_{!_J}$ where $!_J\colon J \to 1$.

Definition

An indexed category has ***indexed products*** if each Δ_r has a right adjoint Π_r and these satisfy the right Beck-Chevalley condition.

Write Π_J instead of $\Pi_{!_J}$ where $!_J\colon J \to 1$.

Set-indexing of a category **C**

$$\prod_{j\in J} \mathbf{C} \;\underset{\Pi_r}{\overset{\Sigma_r}{\rightleftarrows}}\; \prod_{k\in K} \mathbf{C} \qquad \Sigma_r \dashv \Delta_r \dashv \Pi_r, \quad \Delta_r : \prod_{k\in K} \mathbf{C} \longrightarrow \prod_{j\in J} \mathbf{C}$$

$$J \xrightarrow{\quad r \quad} K$$

in **Set**

$$\big(X_j\big)_{j\in J} \overset{\Sigma_r}{\longmapsto} \Big(\sum_{j\in r^{-1}k} X_j\Big)_{k\in K}$$

$$\big(Y_{rj}\big)_{j\in J} \overset{\Delta_r}{\longleftarrow\!\shortmid} \big(Y_k\big)_{k\in K}$$

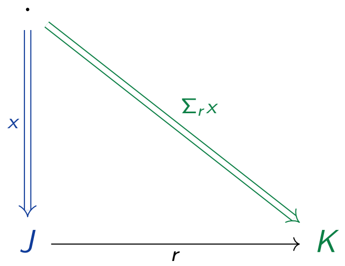
$$\big(X_j\big)_{j\in J} \overset{\Pi_r}{\longmapsto} \Big(\prod_{j\in r^{-1}k} X_j\Big)_{k\in K}$$

Self indexing of a finitely complete category $\mathbf{C}$

$$\mathbf{C}/J \underset{\Delta_r}{\overset{\Sigma_r}{\rightleftarrows}} \mathbf{C}/K \quad (\Sigma_r \dashv \Delta_r)$$

$$J \xrightarrow{\; r \;} K$$

in $\mathbf{C}$



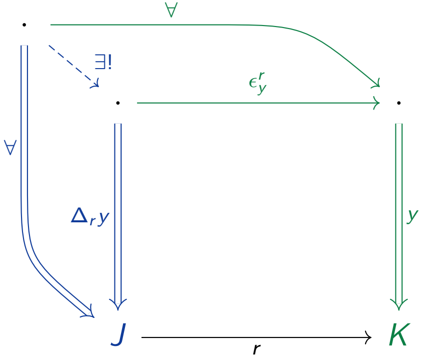
Self indexing of a finitely complete category $\mathbf{C}$

$$\mathbf{C}/J \underset{\Delta_r}{\overset{\Sigma_r}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{C}/K$$

$$J \xrightarrow{\;r\;} K$$

in $\mathbf{C}$

A right adjoint to Σ_r is a choice, for each $(Y, y) \in \mathbf{C}/K$, of $\Delta_r(Y, y) \in \mathbf{C}/J$ and $\epsilon^r_y \colon \Sigma_r \Delta_r(Y, y) \to (Y, y)$ such that $(\Delta_r(Y, y), \epsilon^r_y)$ is terminal in $\Delta_r/(Y, y)$.



Canonicity of self indexing (informally)

Let **C** be a finitely complete category.
Equivalently, **C** is a cartesian monoidal category with equalisers.

The self indexing of **C** seems to be canonical, providing the foundation for

- categories internal to **C**
- dependent lenses/polynomials in **C**
- multivariate polynomial functors in **C**
- models of dependent type theories in **C**

Is there still a canonical indexing of **C** if the monoidal product of **C** is not cartesian?

Outline

Goal

cartesian monoidal category $\mathbf{C}$ with equalisers $\rightsquigarrow$ symmetric monoidal category $\mathcal{V}$ with coreflexive equalisers preserved by all $X \otimes - \otimes Y$

Example

- A partial function $f\colon A \to B$ is a total function $\bar{f}\colon A \to B + 1$.
- Let **Par** denote the category of sets and partial functions.
- The cartesian product on **Set** gives a symmetric monoidal product on **Par**.
- The equaliser of $f, g\colon A \to B$ in **Par** is the equaliser of $\bar{f}, \bar{g}\colon A \to B + 1$ in **Set**, but viewed as a partial function.
- Can check that equalisers in **Par** are preserved by all $X \otimes - \otimes Y$

Goal

cartesian monoidal category $\mathbf{C}$ with equalisers $\rightsquigarrow$ symmetric monoidal category $\mathcal{V}$ with nice coreflexive equalisers

$$\mathbf{C} \rightsquigarrow \mathbf{CoComon}_{\mathcal{V}}$$

Comonoids

A ***(cocommutative) comonoid*** J consists of

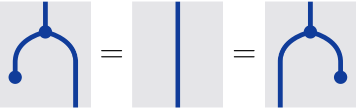
J

object

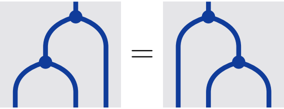
comultiplication

counit

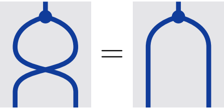
subject to the axioms



counitality



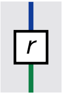
coassociativity



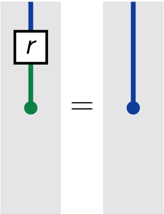
cocommutativity

Comonoid morphisms

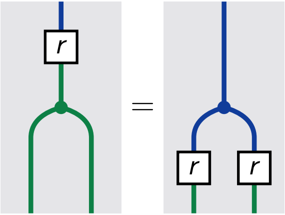
A ***comonoid morphism*** $r \colon J \to K$ consists of



subject to the axioms



preserves counit



preserves comultiplication

Comonoids in a cartesian monoidal category

Let **C** be a cartesian monoidal category.

Proposition

$\mathbf{CoComon_C} \cong \mathbf{C}$

Partial proof.

For a comonoid (J, δ, ϵ) in **C**,

- $\epsilon \colon J \to 1$ is the unique such map,
- $\delta \colon J \to J \times J$ is of the form $\langle \delta_1, \delta_2 \rangle$, and the countitality laws imply that $\delta_1 = \delta_2 = \mathrm{id}_J$. □

Comonoids in **Par**

Proposition

$\mathbf{CoComon}_{\mathbf{Par}} \cong \mathbf{Set}$

Partial proof.

For a comonoid (J, δ, ϵ) in **Par**,

- the counit law $\left(J \xrightarrow{\delta} J \otimes J \xrightarrow{\epsilon \otimes \mathrm{id}_J} I \otimes J \cong J\right) = \left(J \xrightarrow{\mathrm{id}_J} J\right)$ implies that δ is total and so $\delta = \langle \delta_1, \delta_2 \rangle$ in **Set**; it also implies that $\delta_2 = \mathrm{id}_J$.
- the other counit law similarly implies that $\delta_1 = \mathrm{id}_J$, and also that ϵ is total.

Let $r\colon (J, \delta_J, \epsilon_J) \to (K, \delta_K, \epsilon_K)$ be a comonoid morphism in **Par**.

- As ϵ_J is total, the counit preservation law $\left(J \xrightarrow{r} K \xrightarrow{\epsilon_K} I\right) = \left(J \xrightarrow{\epsilon_J} I\right)$ implies that $r\colon J \to K$ is also total. □

Goal

cartesian monoidal category $\mathbf{C}$ with equalisers $\rightsquigarrow$ symmetric monoidal category $\mathcal{V}$ with nice coreflexive equalisers

$$\mathbf{C} \rightsquigarrow \mathbf{CoComon}_{\mathcal{V}}$$

$$\mathbf{C}/J \rightsquigarrow \mathbf{Comod}_{\mathcal{V}}J$$

Comodules

For a comonoid J, a J-***comodule*** (X, x) consists of

object J-coaction

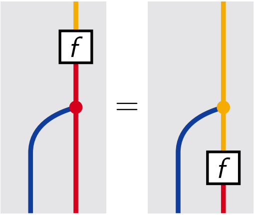
subject to the axioms

counital coassociative

Comodule morphisms

A *J-comodule morphism* $f : (X, x) \to (X', x')$ consists of

subject to the axioms



preserves coaction

Let **C** be a cartesian monoidal category.

Proposition

$\mathbf{Comod}_{\mathbf{C}} J \cong \mathbf{C}/J$

Comodules in **Par**

Proposition

$\mathbf{Comod}_{\mathbf{Par}} J$ *is isomorphic to the category with*

- *objects: pairs* (M, m) *where* M *is a set and* $m\colon M \to J$ *is a total function*
- *morphisms* $(M, m) \to (N, n)$*: partial functions* $f\colon M \to N$ *such that* $n\big(f(x)\big) = m(x)$ *for all* $x \in M$ *on which* f *is defined.*

Proposition

$$\mathbf{Comod}_{\mathbf{Par}} J \cong \prod_{j \in J} \mathbf{Par}$$

Goal

cartesian monoidal category $\mathbf{C}$ with equalisers	$\rightsquigarrow$	symmetric monoidal category $\mathcal{V}$ with nice coreflexive equalisers

$$\mathbf{C} \rightsquigarrow \mathbf{CoComon}_{\mathcal{V}}$$

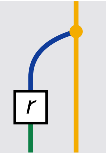
$$\mathbf{C}/J \rightsquigarrow \mathbf{Comod}_{\mathcal{V}} J$$

composition $\rightsquigarrow$ corestriction

Corestriction

The ***corestriction*** of a J-comodule (X, x) along a comonoid morphism $r\colon J \to K$ is the K-comodule

X

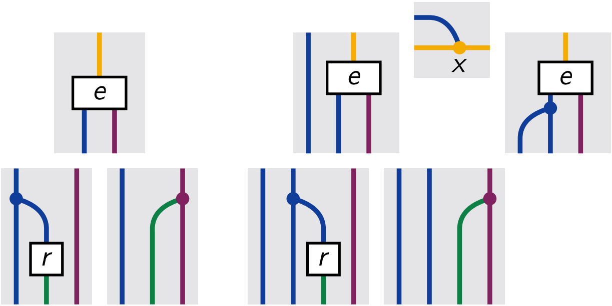


object $\quad$ K-coaction

Goal

cartesian monoidal category $\mathbf{C}$ with equalisers	$\rightsquigarrow$	symmetric monoidal category $\mathcal{V}$ with nice coreflexive equalisers
$\mathbf{C}$	$\rightsquigarrow$	$\mathbf{CoComon}_{\mathcal{V}}$
$\mathbf{C}/J$	$\rightsquigarrow$	$\mathbf{Comod}_{\mathcal{V}}J$
composition	$\rightsquigarrow$	corestriction
pullback	$\rightsquigarrow$	coinduction

The ***coinduction*** of a K-comodule (Y, y) along a comonoid morphism $r \colon X \to Y$ is the J-comodule (X, x) given by the coreflexive equalisers



Goal

cartesian monoidal category **C** with equalisers	$\rightsquigarrow$	symmetric monoidal category $\mathcal{V}$ with nice coreflexive equalisers
C	$\rightsquigarrow$	$\mathbf{CoComon}_{\mathcal{V}}$
$\mathbf{C}/J$	$\rightsquigarrow$	$\mathbf{Comod}_{\mathcal{V}}J$
composition	$\rightsquigarrow$	corestriction
pullback	$\rightsquigarrow$	coinduction

Outline

1. Introduction to indexed category theory
2. Comonoid indexing of nice symmetric monoidal categories
3. Universality of self indexing and comonoid indexing

Universality of self indexing

Proposition

There is an adjunction

$$\mathbf{SomeIndCat} \underset{\textit{self indexing}}{\overset{\textit{underlying category}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CompCat}$$

(up to unique natural isomorphisms).

Idea

For an $\mathbf{S}$-indexed category $\mathbb{C}$ with the necessary properties, the adjunction has unit with $\mathbb{C}$-component $\mathbb{C} \to \mathbb{S}\mathbf{elf}(\mathbf{C}^1)$ given by the functor $F\colon \mathbf{S} \to \mathbf{C}^1$

$$
\begin{array}{ccc}
J & \xrightarrow{\quad r \quad} & K \\
\downarrow & & \downarrow \\
\Sigma_J \Delta_J 1 \cong \Sigma_K \Sigma_r \Delta_r \Delta_K 1 & \xrightarrow{\Sigma_K \epsilon^r \Delta_K 1} & \Sigma_K \Delta_K 1
\end{array}
$$

and the functors $G^J\colon \mathbf{C}^J \to \mathbf{C}^1/FJ$

$$
\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & N \\
\downarrow & & \downarrow \\
(\Sigma_J M, \Sigma_J !_M) & \xrightarrow{\Sigma_J f} & (\Sigma_J N, \Sigma_J !_N)
\end{array}
$$

Universality of comonoid indexing

Conjecture

There is an adjunction

$$\textbf{SomeIndSymMonCat} \underset{\textit{comonoid indexing}}{\overset{\textit{underlying category}}{\rightleftarrows}} \textbf{EqSymMonCat} \quad (\bot)$$

(up to unique natural isomorphisms).

For an **S**-indexed symmetric monoidal category $\mathbb{V}$ with the necessary properties, the adjunction has unit with $\mathbb{V}$-component $\mathbb{V} \to \mathbb{C}\mathbf{o}\mathbb{C}\mathbf{omon}(\mathcal{V}^1)$ given by the functor $F\colon \mathbf{S} \to \mathcal{V}^1$

$$\begin{array}{ccc} J & \xrightarrow{\quad r \quad} & K \\ \downarrow & & \downarrow \\ \Sigma_J \Delta_J I \cong \Sigma_K \Sigma_r \Delta_r \Delta_K I & \xrightarrow{\Sigma_K \epsilon^r \Delta_K I} & \Sigma_K \Delta_K I \end{array}$$

and the functors $G^J\colon \mathcal{V}^J \to \mathbf{Comod}_{\mathcal{V}^I} FJ$

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ \downarrow & & \downarrow \\ (\Sigma_J M, \Sigma_J \eta^J_M) & \xrightarrow{\Sigma_J f} & (\Sigma_J N, \Sigma_J \eta^J_N) \end{array}$$

Conclusion

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- The comonoid indexing deserves to be better known
- The self indexing and comonoid indexing satisfy similar universal properties

Next steps

- Work out the details of the comonoid indexing universal property
- Find even more examples of suitable monoidal categories
- Links with linear dependent type theory or linear dependent lenses?