Abstract Contractions

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(Joint work with Chris Heunen)

Edinburgh Category Theory Seminar
October 2023

Recap of Hilbert Spaces and Dagger Categories

A Hilbert space is a vector space X equipped with an inner product $\langle - | - \rangle : X \times X \to \mathbb{C}$ whose associated norm is Cauchy complete.

A linear map $f : X \to Y$ is bounded if there is a $C > 0$ such that $\|fx\| \leqslant C\|x\|$.

The Hermitian adjoint of a bounded linear map $f : X \to Y$ is the bounded linear map $f^\dagger : Y \to X$ where $f^\dagger y$ is uniquely determined by the equation $\langle fx | y \rangle = \langle x | f^\dagger y \rangle$.

A dagger category is a category $\underline{C}$ equipped with a functor

$$(-)^{\dagger} : \underline{C}^{op} \to \underline{C}$$

such that $X^{\dagger} = X$ and $f^{\dagger\dagger} = f$.

Let $\underline{\text{Hilb}}$ denote the dagger category of Hilbert spaces and bounded linear maps.

The Hilbert space $0 = \{0\}$
is a zero object of $\underline{\mathrm{Hilb}}$.

The zero morphism $0_{X,Y} : X \longrightarrow Y$
is given by $0_{X,Y}\, x = 0$.

A bounded linear map $f : X \to Y$ is an isometry if

$$\|fx\| = \|x\|.$$

A bounded linear map $f : X \to Y$ is an isometry if and only if

$$f^\dagger f = 1_X.$$

Isometries represent closed subspaces.

A morphism $f : X \to Y$ is dagger monic if $f^\dagger f = 1_X$.

In diagrams:

$\rightarrowtail$ is dagger monic

$\twoheadrightarrow$ is dagger epic

The dagger monos in $\underline{\mathrm{Hilb}}$ are the isometries.

Let $\underline{\mathrm{Mono}}^\dagger(\underline{C})$ be the wide subcategory of dagger monos in $\underline{C}$.

The kernel of a bounded linear map $f : X \to Y$ is the subspace

$$\operatorname{Ker} f = \{x \in X : fx = 0\}$$

of X with the restriction of the inner product of X.

The canonical inclusion $\operatorname{Ker} f \hookrightarrow X$ is an isometry.

A dagger kernel of a morphism $f : X \to Y$ is a dagger monic equaliser of f and $0_{X,Y}$.

In Hilb, every morphism has a dagger kernel.

The direct sum of Hilbert spaces X and Y is the space

$$X \oplus Y = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in X, y \in Y \right\}$$

with the inner product

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle = \langle x | x' \rangle + \langle y | y' \rangle.$$

The canonical injections

$i_1 : X \to X \oplus Y$ and $i_2 : Y \to X \oplus Y$

$i_1 x = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\qquad$ $i_2 y = \begin{pmatrix} 0 \\ y \end{pmatrix}$

are orthogonal isometries.

A dagger coproduct of objects X and Y is a coproduct $(X \oplus Y, i_1, i_2)$ of X and Y such that i_1 and i_2 are orthogonal and dagger monic.

All dagger coproducts are biproducts where

$$p_1 = i_1^\dagger \quad \text{and} \quad p_2 = i_2^\dagger.$$

In Hilb, every pair of objects has a dagger coproduct.

Ultrarational Dagger Categories

Want simple category theoretic axioms that capture the fundamental properties of <u>Hilb</u>.

Similar to abelian categories and <u>Ab</u>.

DEFINITION: A dagger category $\underline{C}$ is *ultrarational* if

(U1) it has a zero object,

(U2) every pair of objects has a dagger coproduct,

(U3) every morphism has a dagger kernel, and

(U4) every diagonal morphism $\Delta : X \to X \oplus X$ is a kernel.

$\underline{\text{Hilb}}$ and $\underline{\text{FHilb}}$ are ultrarational.

Properties of ultrarational dagger categories:

- Uniquely enriched in $\underline{\mathrm{Vect}}_{\mathbb{Q}}$.
- Have finite dagger equalisers.
- Have (epi, dagger mono) factorisations and are regular.
- Every dagger mono is a dagger coproduct injection.
  (has an orthogonal complement)
- Each endomorphism algebra is canonically partially ordered.

RATIONAL DAGGER CATEGORIES

MATTHEW DI MEGLIO

ABSTRACT. The notion of *abelian category* is an elegant distillation of the fundamental properties of the category of abelian groups, comprising a few simple axioms about products and kernels. Whilst the categories of real, complex, and quaternionic Hilbert spaces and bounded linear maps are not abelian, they satisfy almost all of the axioms. Heunen's notion of *Hilbert category* is an attempt at adapting the abelian-category axioms to capture instead the essence of these categories of Hilbert spaces. The key idea is to encode Hermitian adjoints with a *dagger*—an identity-on-objects involutive contravariant endofunctor. One limitation is the symmetric monoidal structure, which is used to construct additive inverses of morphisms; such additional structure is not needed for the analagous result about abelian categories, and it excludes non-commutative examples like the dagger category of quaternionic Hilbert spaces.

This article introduces the notion of *rational dagger category*—a successor to the notion of Hilbert category whose theory is closer to that of abelian categories. In particular, a monoidal product is not required. They are named after their enrichment in the category of rational vector spaces. Whilst the dagger categories of real, complex, and quaternionic Hilbert spaces are the motivating examples, others include the dagger categories of matrices over a formally-complex involutive division ring and of finite-dimensional inner-product spaces over a semiordered involutive division ring. Also introduced are the notion of *orthogonally complemented dagger category*, whose axioms capture the connection between orthogonal complements and coproducts, and generalisations of the notions of dagger monomorphism and dagger product, called *dagger section* and *orthogonal product*, which play an important role in the theory of both orthogonally complemented and rational dagger categories.

Available soon at

https://mdimeglio.github.io

Contractions

A linear map $f: X \to Y$ is a **contraction** if $\|fx\| \leqslant \|x\|$.

A **cospan dilation** (a_1, A, a_2) of a contraction $f: X \to Y$ is a cospan

$$X \xrightarrow{a_1} A \xleftarrow{a_2} Y$$

of isometries such that $a_2^\dagger a_1 = f$.

$$X \xrightarrow{\begin{pmatrix}\sqrt{1_X - f^\dagger f} \\ f\end{pmatrix}} X \oplus Y \xleftarrow{\begin{pmatrix}0 \\ 1\end{pmatrix}} Y$$

is a cospan dilation of $f: X \to Y$.

DEFINITION:

In a dagger category $\underline{\mathbb{C}}$ with dagger biproducts, $f: X \to Y$ is a **contraction** if $f = a_2^\dagger a_1$ for some cospan

$$X \overset{a_1}{\rightarrowtail} A \overset{a_2}{\leftarrowtail} Y$$

of dagger biproduct injections.

Equivalently exists unitary $u: X \oplus R \to Y \oplus S$

$$\begin{tikzcd} X \oplus R \arrow[r, "u"] & Y \oplus S \arrow[d, "p_1"] \\ X \arrow[u, "i_1"] \arrow[r, "f"'] & Y \end{tikzcd}$$

PROPOSITION: Contractions form a wide dagger subcategory $\underline{\mathrm{Con}(\underline{C})}$ of $\underline{C}$.

$$
\begin{tikzcd}
 & & Z\oplus T\oplus S \arrow[d, "p_{1,2}"] \\
 & Y\oplus S \arrow[ur, "b\oplus 1_S"] \arrow[d, "p_1"] & Z\oplus T \arrow[d, "p_1"] \\
X \arrow[ur, "a"] \arrow[r, "f"'] & Y \arrow[ur, "b"] \arrow[r, "g"'] & Z
\end{tikzcd}
$$

PROPOSITION: In an ultrarational dagger category, $f : X \to Y$ is a contraction if and only if $f^\dagger f \leqslant 1_X$.

$f^\dagger f \leqslant 1_X$

$\Updownarrow$

exists $s : X \to S$ such that $f^\dagger f + s^\dagger s = 1_X$

$\Leftrightarrow$

exists $s : X \to S$ such that $\langle f, s\rangle^\dagger \langle f, s\rangle = 1_X$

$\Updownarrow$

exists $a : X \longrightarrow Y \oplus S$ such that $a^\dagger a = 1_X$ and $p_1 a = f$

Directed Colimits of Contractions

In Heunen and Kornell's characterisation of $\underline{\mathbf{Hilb}}$, directed colimits in $\underline{\mathbf{Mono}^{\dagger}}(\underline{\mathbb{C}})$ encoded completeness.

$$
\begin{tikzcd}
I \arrow[r, tail, "i_1"] \arrow[rrrrr, tail, bend right, "j_1"'] & I^{\oplus 2} \arrow[r, tail, "i_{1,2}"] \arrow[rrrr, tail, bend right, "j_2"'] & I^{\oplus 3} \arrow[r, tail, "i_{1,2,3}"] \arrow[rrr, tail, bend right, "j_3"'] & I^{\oplus 4} \arrow[r] \arrow[rr, tail, bend right, "j_4"'] & \cdots & I^{\oplus \mathbb{N}}
\end{tikzcd}
$$

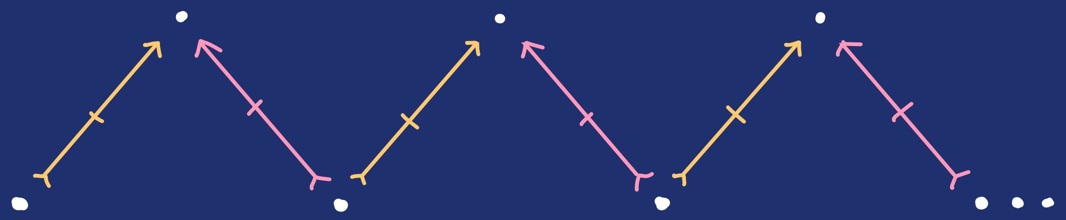
Obtained indirectly via Solèr's theorem

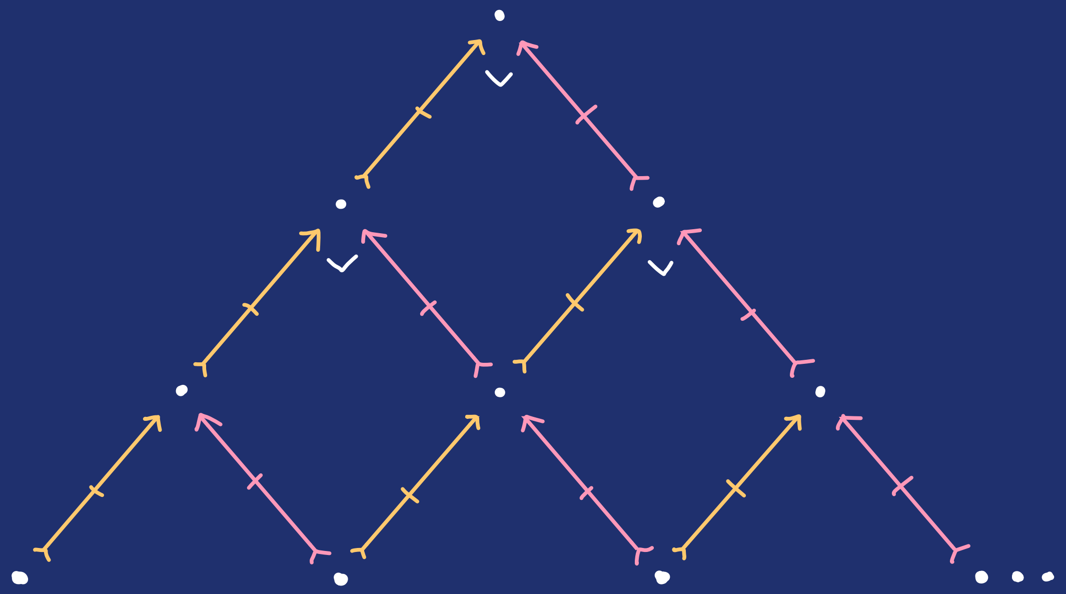
In my last talk here, I showed how to construct infima of decreasing sequences of positive scalars using directed colimits in $\underline{\mathrm{Con}}(\underline{C})$.

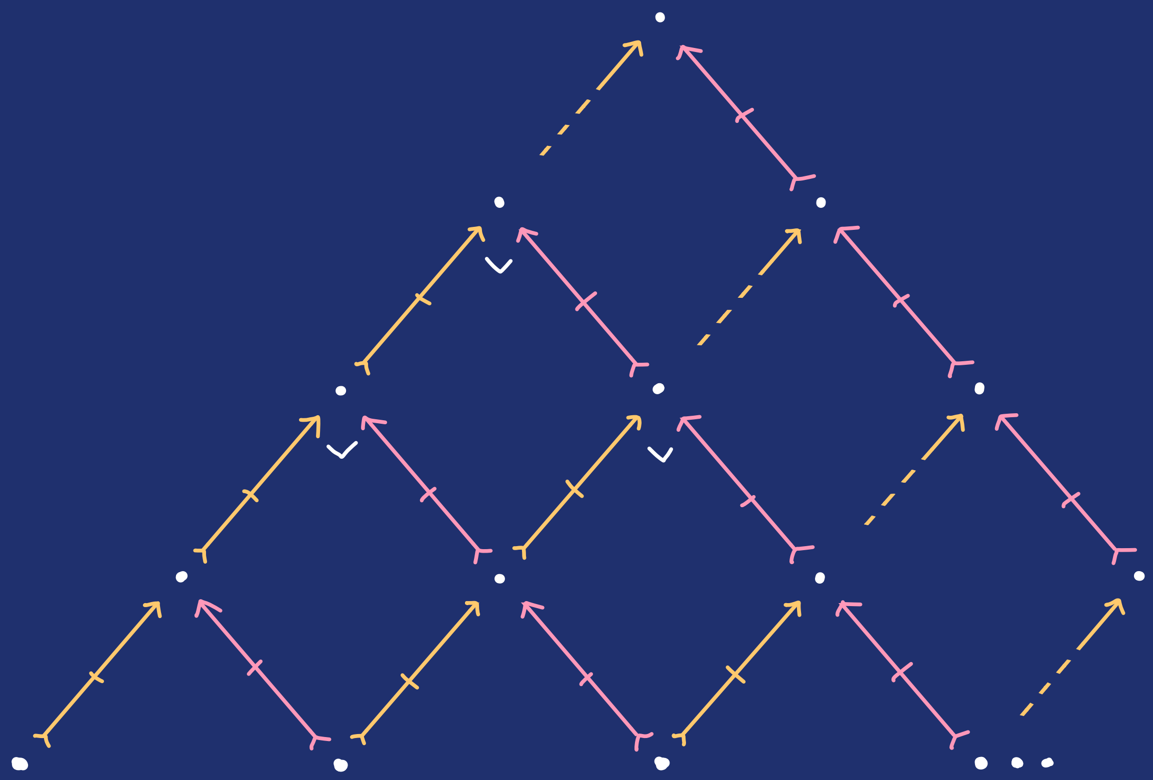
$$
\begin{tikzcd}
 & I \arrow[dl, bend right, "x_1"'] \arrow[d, "x_2"] \arrow[dr, bend left, "x_3"] \arrow[ddrr, bend left, "x"] & & \\
X_1 \arrow[r, "x_2x_1^{-1}"'] \arrow[drrr, bend right, "xx_1^{-1}"'] & X_2 \arrow[r, "x_3x_2^{-1}"'] \arrow[drr, bend right, "xx_2^{-1}"'] & X_3 \arrow[r] \arrow[dr, bend right, "xx_3^{-1}"'] & \cdots \\
 & & & X
\end{tikzcd}
$$

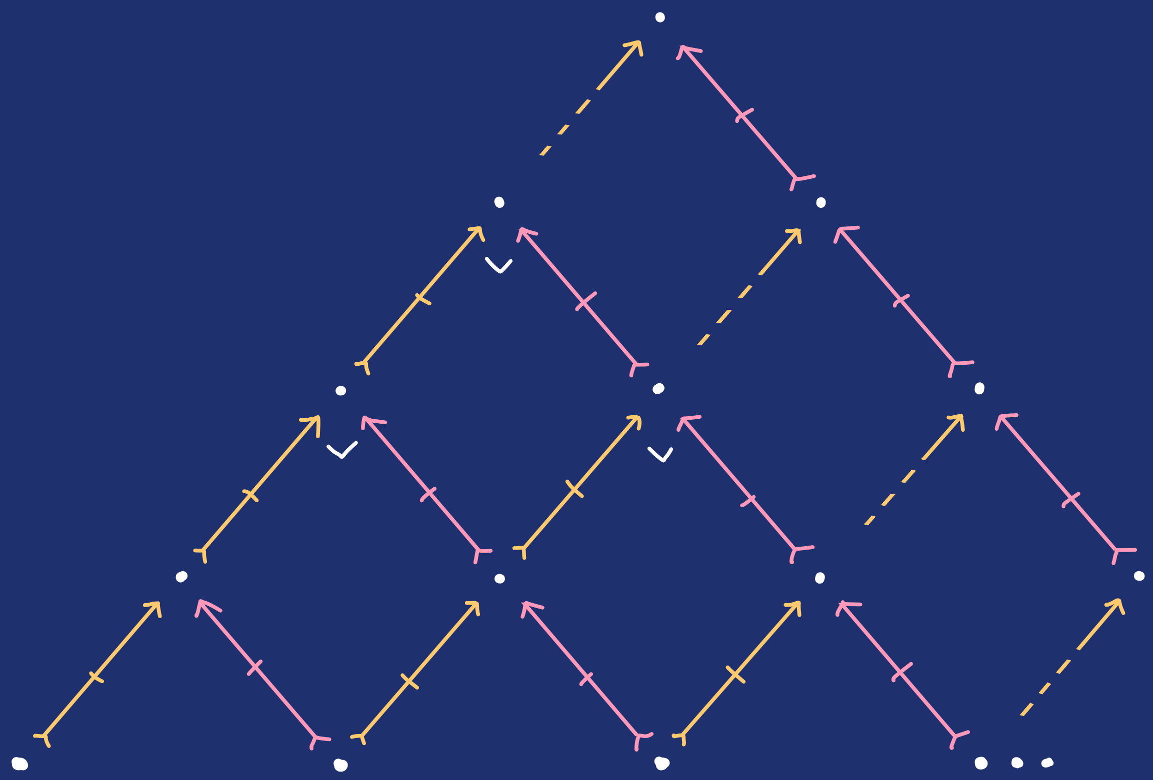
$$x_1^\dagger x_1 \geqslant x_2^\dagger x_2 \geqslant x_3^\dagger x_3 \geqslant x^\dagger x = \inf x_n^\dagger x_n$$

Can we construct directed colimits in $\underline{\mathrm{Con}}(\underline{\mathcal{C}})$ from those in $\mathrm{Mono}^{+}(\underline{\mathcal{C}})$?

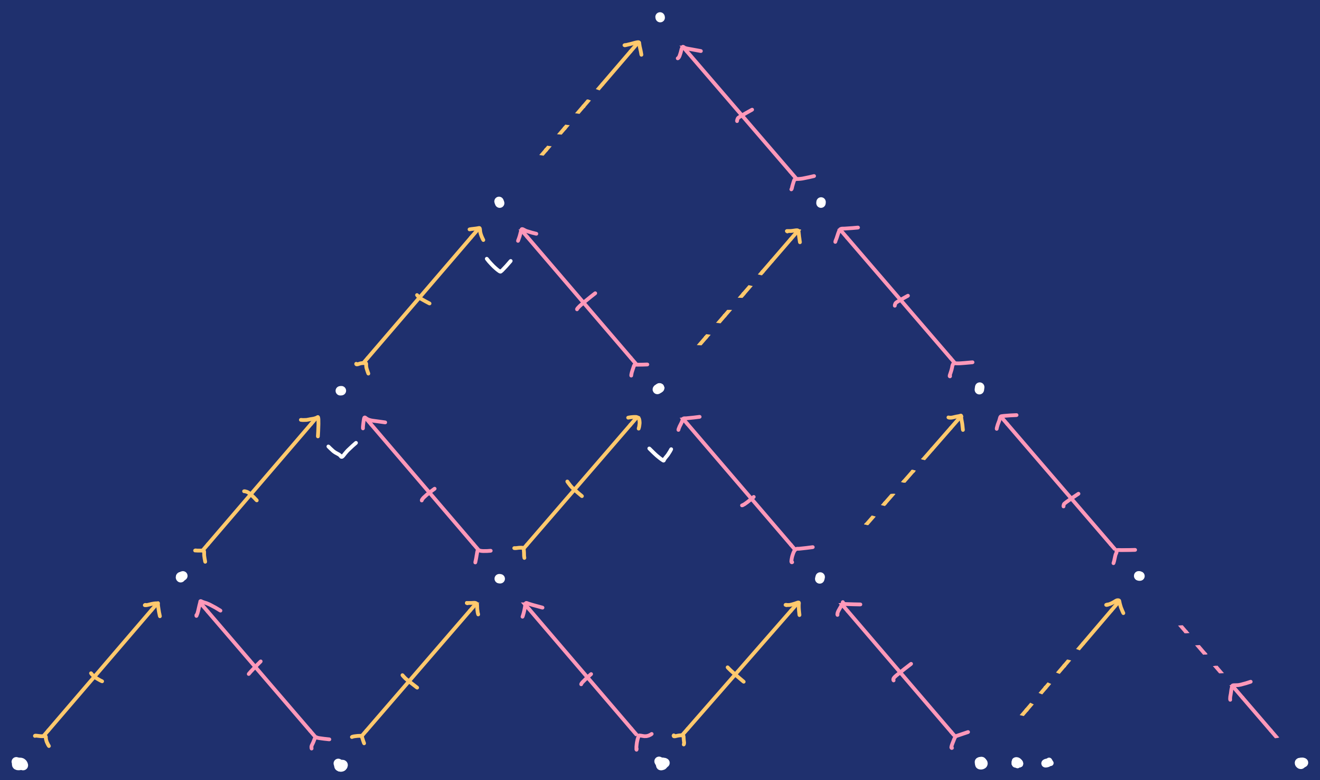






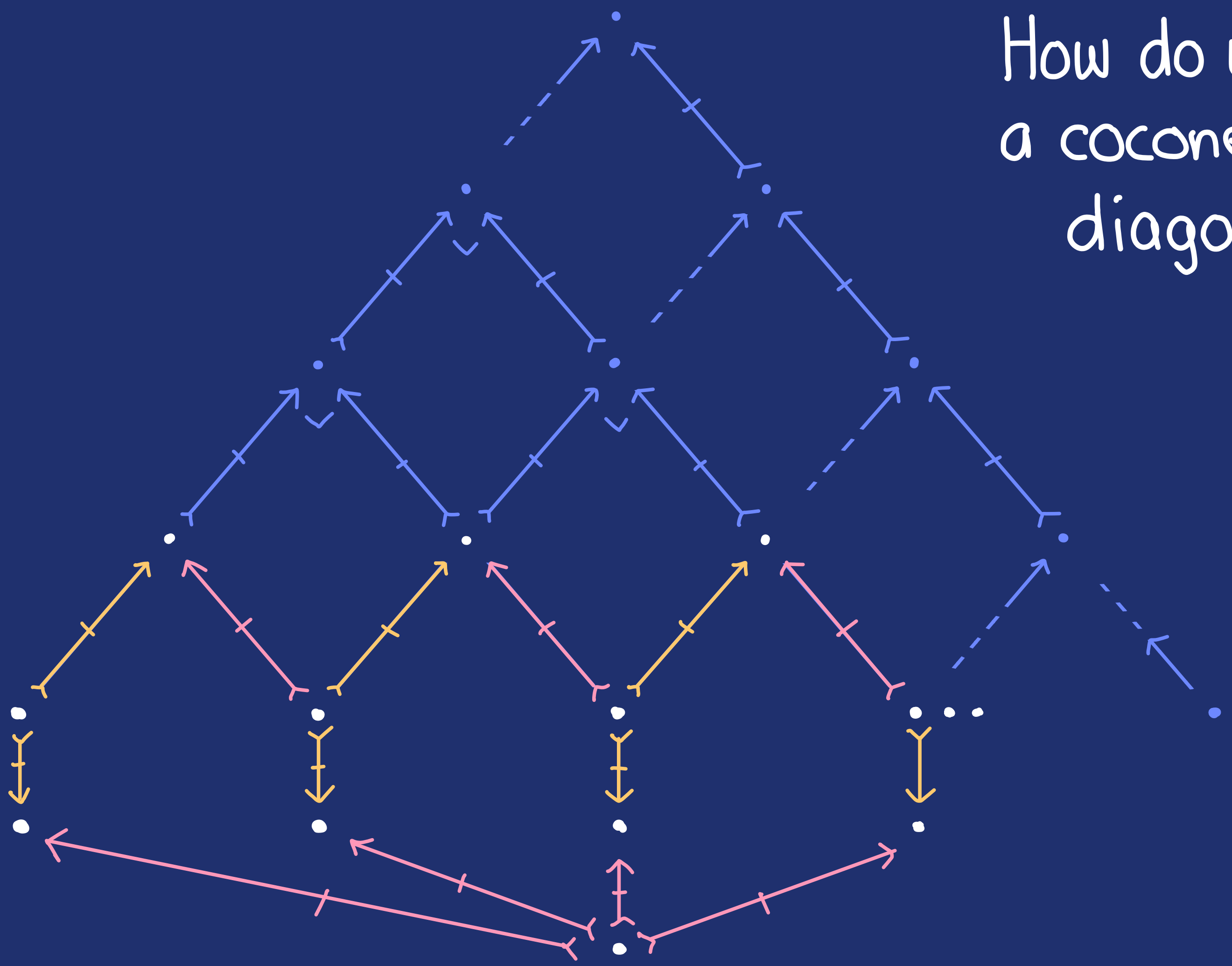


$$\underline{\mathrm{Mono}}^{\dagger}(\underline{C})/X \xrightarrow[\sim]{(-)^{\perp}} \left(\underline{\mathrm{Mono}}^{\dagger}(\underline{C})/X\right)^{\mathrm{op}}$$

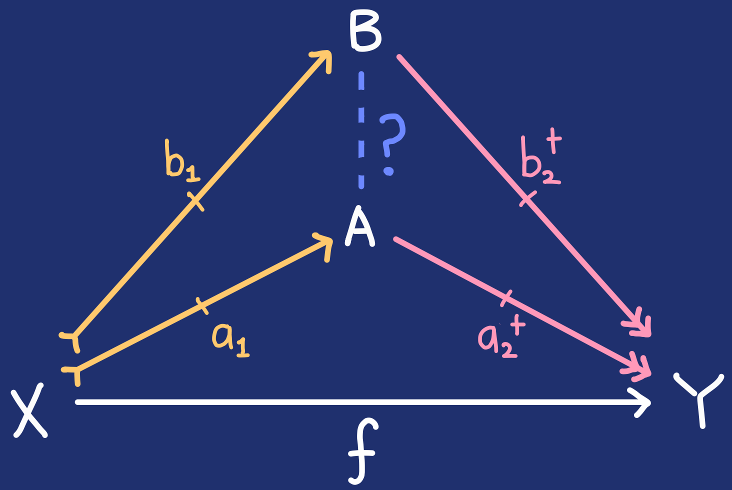


$$\underline{\mathrm{Mono}}^{\dagger}(\underline{C})/X \xrightarrow[\sim]{(-)^{\perp}} \left(\underline{\mathrm{Mono}}^{\dagger}(\underline{C})/X\right)^{\mathrm{op}}$$

How do we form a cocone on the diagonals?



Need a way to relate different cospan dilations of the same contraction



DEFINITION:

A cospan dilation (a_1, A, a_2) of a contraction $f : X \to Y$ is minimal if for all cospan dilations (b_1, B, b_2), there is a unique dagger mono $m : A \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
& B & \\
& A \arrow[u, dashed, "m"] & \\
X \arrow[uur, "b_1"] \arrow[ur, "a_1"'] & & Y \arrow[uul, "b_2"'] \arrow[ul, "a_2"]
\end{tikzcd}$$

LEMMA: In an ultrarational dagger category, every contraction has a jointly epic cospan dilation.

$$\begin{tikzcd} & B & \\ X \arrow[ur, tail, "b_1"] & & Y \arrow[ul, tail, "b_2"'] \end{tikzcd}$$

LEMMA: In an ultrarational dagger category, every contraction has a jointly epic cospan dilation.

$$\begin{tikzcd} & & B & & \\ X \arrow[urr, "b_1"] \arrow[rr, tail, "i_1"'] & & X \oplus Y \arrow[u, bend left, "{[b_1\ b_2]}"] & & Y \arrow[ull, "b_2"'] \arrow[ll, tail, "i_2"] \end{tikzcd}$$

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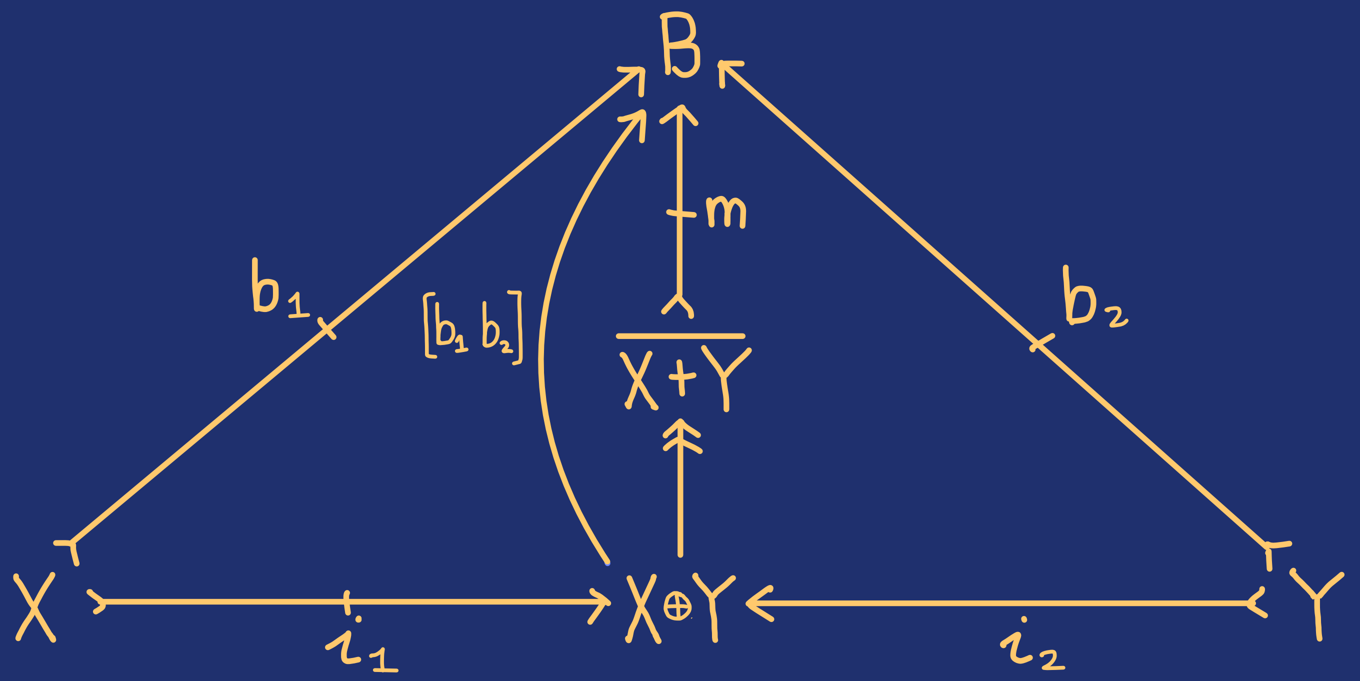

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$$
\begin{tikzcd}
 & & B & & \\
 & & \overline{X+Y} \arrow[u, tail, "m"'] & & \\
X \arrow[uurr, tail, "b_1"] \arrow[urr, tail, "a_1"] \arrow[rr, tail, "i_1"'] & & X \oplus Y \arrow[u, two heads] \arrow[uu, bend left, "{[b_1\ b_2]}"] & & Y \arrow[uull, tail, "b_2"'] \arrow[ull, tail, "a_2"'] \arrow[ll, tail, "i_2"]
\end{tikzcd}
$$

LEMMA: In Hilb, jointly epic cospan dilations are minimal.

$$
\begin{tikzcd}
 & B & \\
 & A & \\
X \arrow[uur, tail, "b_1"] \arrow[ur, tail, "a_1"'] & & Y \arrow[uul, tail, "b_2"'] \arrow[ul, tail, "a_2"]
\end{tikzcd}
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$$
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 & & A & & \\
X \arrow[uurr, "b_1"] \arrow[urr, "a_1"] \arrow[rr, "i_1"'] & & X \oplus Y \arrow[u, "{[a_1\ a_2]}"'] \arrow[uu, bend left, "{[b_1\ b_2]}"] & & Y \arrow[uull, "b_2"'] \arrow[ull, "a_2"'] \arrow[ll, "i_2"]
\end{tikzcd}
$$

has dense image

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X \arrow[uur, tail, "b_1"] \arrow[ur, tail, "a_1"] \arrow[r, tail, "i_1"'] & X \oplus Y \arrow[u, two heads, "{[a_1\ a_2]}"'] \arrow[uu, bend left, "{[b_1\ b_2]}"] & Y \arrow[uul, tail, "b_2"'] \arrow[ul, tail, "a_2"'] \arrow[l, tail, "i_2"]
\end{tikzcd}
$$

has dense image

LEMMA: In Hilb, jointly epic cospan dilations are minimal.

m is continuous linear extension of

$$[a_1\ a_2]\begin{bmatrix} x \\ y \end{bmatrix} \mapsto [b_1\ b_2]\begin{bmatrix} x \\ y \end{bmatrix}$$

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\end{tikzcd}$$

has dense image

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X \arrow[uurr, "b_1"] \arrow[urr, "a_1"'] \arrow[rr, "i_1"'] & & X \oplus Y \arrow[u, "{[a_1\ a_2]}"'] \arrow[uu, bend left, "{[b_1\ b_2]}"] & & Y \arrow[uull, "b_2"'] \arrow[ull, "a_2"] \arrow[ll, "i_2"]
\end{tikzcd}$$

$[a_1\ a_2]$ has dense image

Well defined because

$$[b_1\ b_2]^\dagger [b_1\ b_2] = \begin{bmatrix} b_1^\dagger b_1 & b_1^\dagger b_2 \\ b_2^\dagger b_1 & b_2^\dagger b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1_X & f^\dagger \\ f & 1_Y \end{bmatrix}$$

$$= [a_1\ a_2]^\dagger [a_1\ a_2]$$

HOPE (Part of Douglas' lemma):

In a sufficiently nice ultrarational dagger category,

$$\begin{tikzcd} X \arrow[drr, dashed, tail, "\exists g"] & & \\ & & B \\ A \arrow[uu, two heads, "e"] \arrow[urr, "\forall f"'] & & \end{tikzcd}$$

$$f^\dagger f = e^\dagger e$$

The dagger mono g is called the extension of f along e.
It is unique because e is epic.

RATIONAL DAGGER CATEGORIES

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ABSTRACT. The notion of *abelian category* is an elegant distillation of the fundamental properties of the category of abelian groups, comprising a few simple axioms about products and kernels. Whilst the categories of real, complex, and quaternionic Hilbert spaces and bounded linear maps are not abelian, they satisfy almost all of the axioms. Heunen's notion of *Hilbert category* is an attempt at adapting the abelian-category axioms to capture instead the essence of these categories of Hilbert spaces. The key idea is to encode Hermitian adjoints with a *dagger*—an identity-on-objects involutive contravariant endofunctor. One limitation is the symmetric monoidal structure, which is used to construct additive inverses of morphisms; such additional structure is not needed for the analagous result about abelian categories, and it excludes non-commutative examples like the dagger category of quaternionic Hilbert spaces.

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