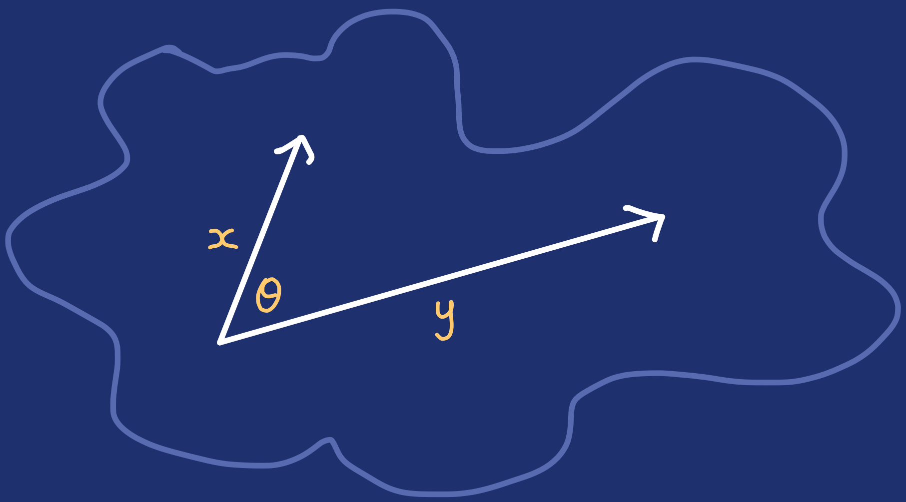
Minimal Dilations Categorically

Matthew Di Meglio

ITACA Fest
September 2024

Hilbert spaces are vector spaces with geometry (encoded by a complete innerproduct)



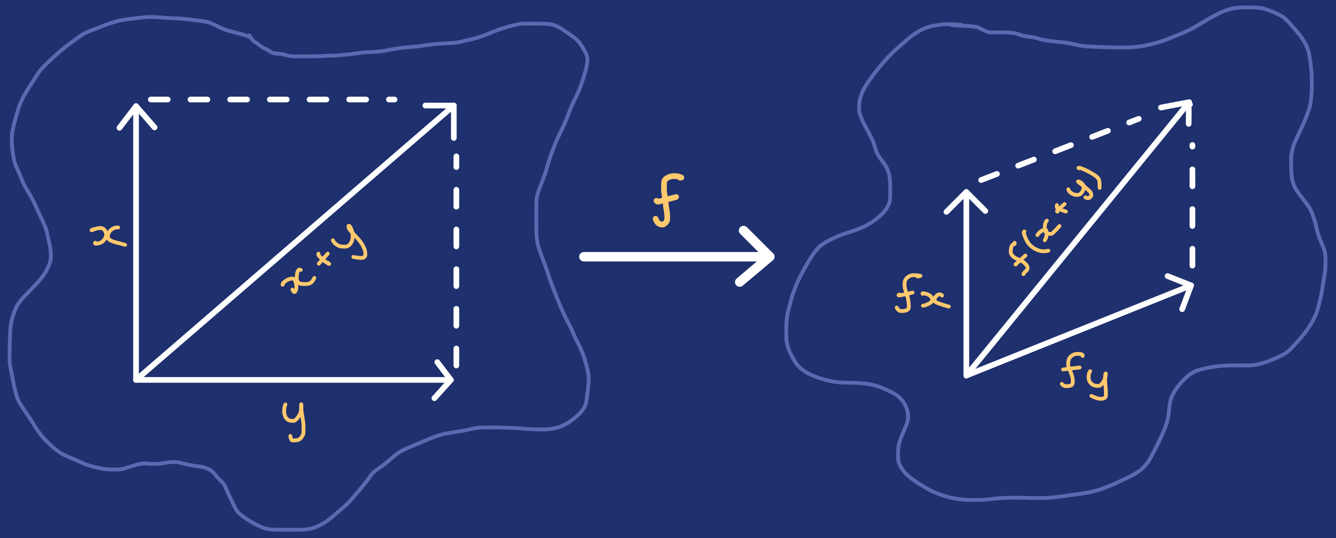
$\|x\| = \sqrt{\langle x | x \rangle}$

Lengths

$\cos\theta = \dfrac{\mathrm{Re}\langle x | y \rangle}{\|x\| \, \|y\|}$

Angles

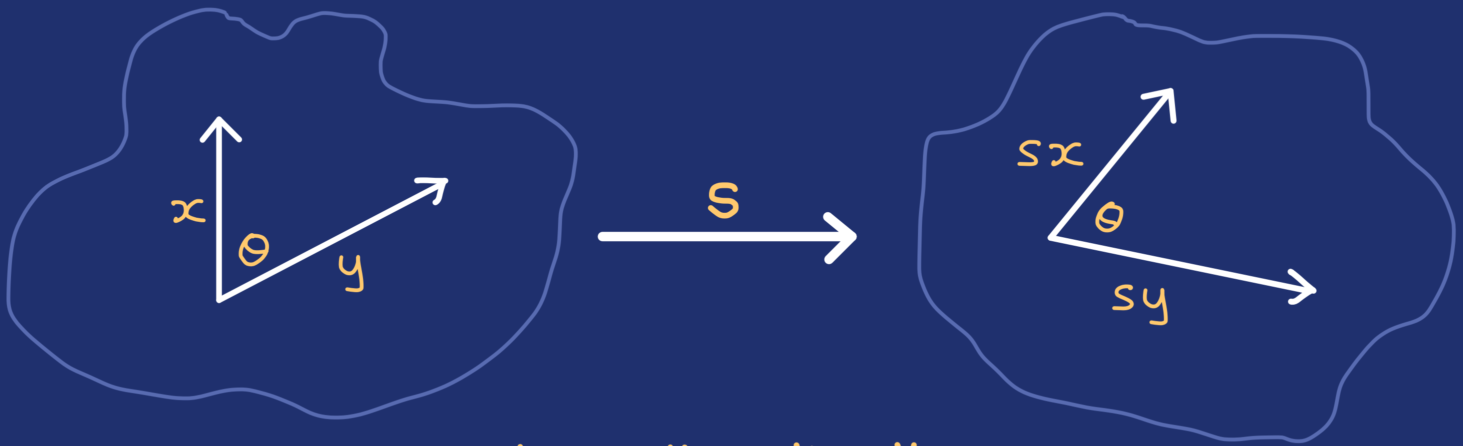
Contractions are linear maps between Hilbert spaces that decrease lengths



$$\|fx\| \leq \|x\|$$

They form a category $\underline{\mathbf{Hilb}_{\leq 1}}$

Isometries are maps between Hilbert spaces that preserve geometry



$$\|sx\| = \|x\|$$

They form a full subcategory $\underline{\mathrm{Hilb}}_1$ of $\underline{\mathrm{Hilb}}_{\leq 1}$

Sz. Nagy's unitary dilation theorem
expresses contractions in terms of
isometries and unitaries

It is the foundation of
the modern theory of contractions

Every contraction $f: X \to X$ has a minimal unitary dilation $u: S \to S$

$$\begin{tikzcd}
X \arrow[r, tail, "s"'] \arrow[d, "f^n"'] & S \arrow[d, "u^n", "\wr"'] \\
X & S \arrow[l, two heads, "s^*"']
\end{tikzcd}$$

$$\boxed{S = \bigvee_{n=-\infty}^{\infty} u^n S}$$

Every contraction $f\colon X \to X$ has a minimal unitary dilation $u\colon S \to S$

$$
\begin{tikzcd}
X \arrow[r, tail, "s"'] \arrow[rr, tail, bend left, "t"] \arrow[d, "f^n"'] & S \arrow[r, tail, dashed, "m"'] \arrow[d, "\wr", "u^n"'] & T \arrow[d, "\wr", "v^n"'] \\
X & S \arrow[l, two heads, "s^*"'] & T \arrow[l, two heads, dashed, "m^*"'] \arrow[ll, two heads, bend left, "t^*"]
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$$\begin{tikzcd}
X \arrow[r, tail, "s"', "|" marking] \arrow[rr, tail, bend left, "t", "|" marking] \arrow[d, "f"'] & S \arrow[r, tail, dashed, "m"', "|" marking] \arrow[d, "u", "\wr"'] & T \arrow[d, "v", "\wr"'] \arrow[ll, two heads, bend left, "t^*", "|" marking] \arrow[l, two heads, dashed, "m^*"', "|" marking] \\
X & S \arrow[l, two heads, "s^*"', "|" marking] & T
\end{tikzcd}$$

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$$
\begin{tikzcd}
X \arrow[r, tail, "s_1"'] \arrow[rr, bend left, tail, "t_1"] \arrow[d, "f"'] & S \arrow[r, tail, dashed, "m"'] \arrow[d, "1"] & T \arrow[d, "1"] \\
X & S \arrow[l, two heads, "s_2^*"'] & T \arrow[l, dashed, two heads, "m^*"'] \arrow[ll, bend left, two heads, "t_2^*"]
\end{tikzcd}
$$

$$\begin{tikzcd}
X_1 \arrow[r, tail, "s_1"'] \arrow[rr, tail, bend left, "t_1"] \arrow[d, "f"'] & S \arrow[r, tail, dashed, "m"'] \arrow[d, "1"] & T \arrow[d, "1"] \\
X_2 & S \arrow[l, two heads, "s_2^*"'] & T \arrow[l, two heads, dashed, "m^*"'] \arrow[ll, two heads, bend left, "t_2^*"]
\end{tikzcd}$$

$\boxed{s_2^* s_1 = f}$ $\boxed{t_2^* t_1 = f}$

$$
\begin{tikzcd}
X_1 \arrow[d, "f"'] \arrow[dr, tail, "s_1"'] \arrow[drr, tail, bend left, "\forall t_1"] & & \\
& S \arrow[r, tail, dashed, "\exists ! m"'] & T \\
X_2 \arrow[ur, tail, "s_2"] \arrow[urr, tail, bend right, "\forall t_2"'] & &
\end{tikzcd}
$$

Call (S, s_1, s_2) a codilator of f

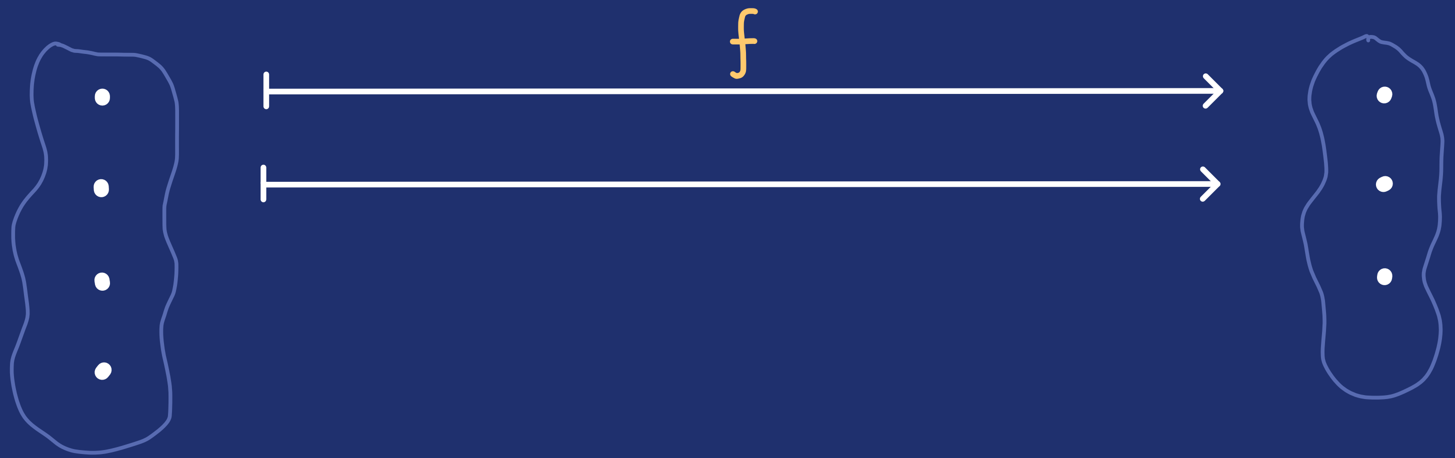
Codilators make sense in the abstract setting of $*$-categories

$1^* = 1$ $\qquad$ $(gf)^* = f^* g^*$ $\qquad$ $(f^*)^* = f$

A morphism $f : X \to Y$ is isometric if $f^* f = 1$

Every morphism in $\underline{\mathrm{Hilb}}_{\leqslant 1}$ has a codilator

A partial bijection is an injective partial function



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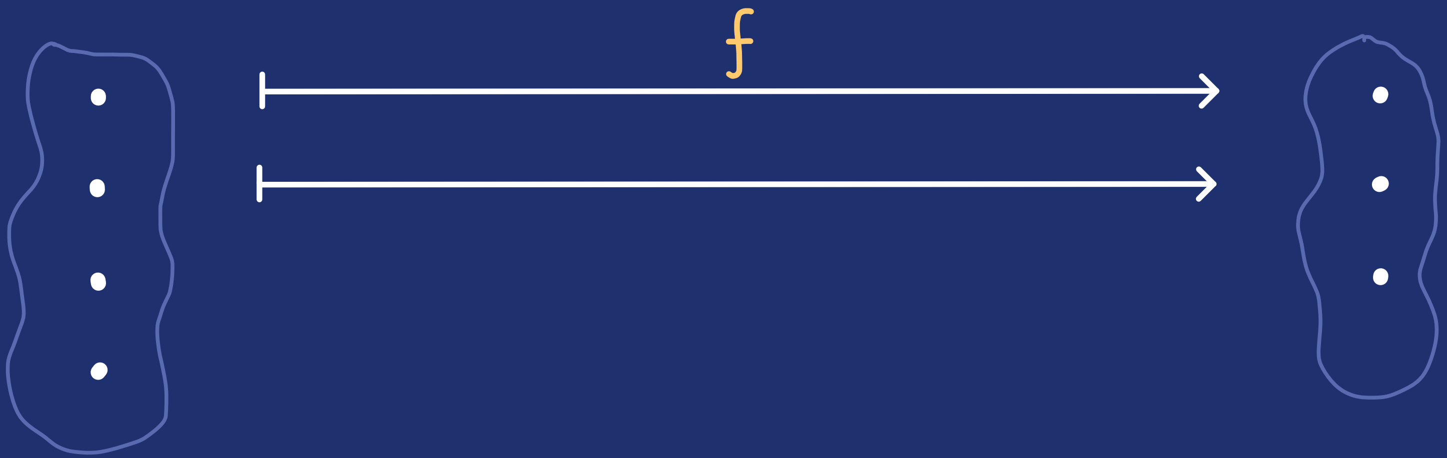

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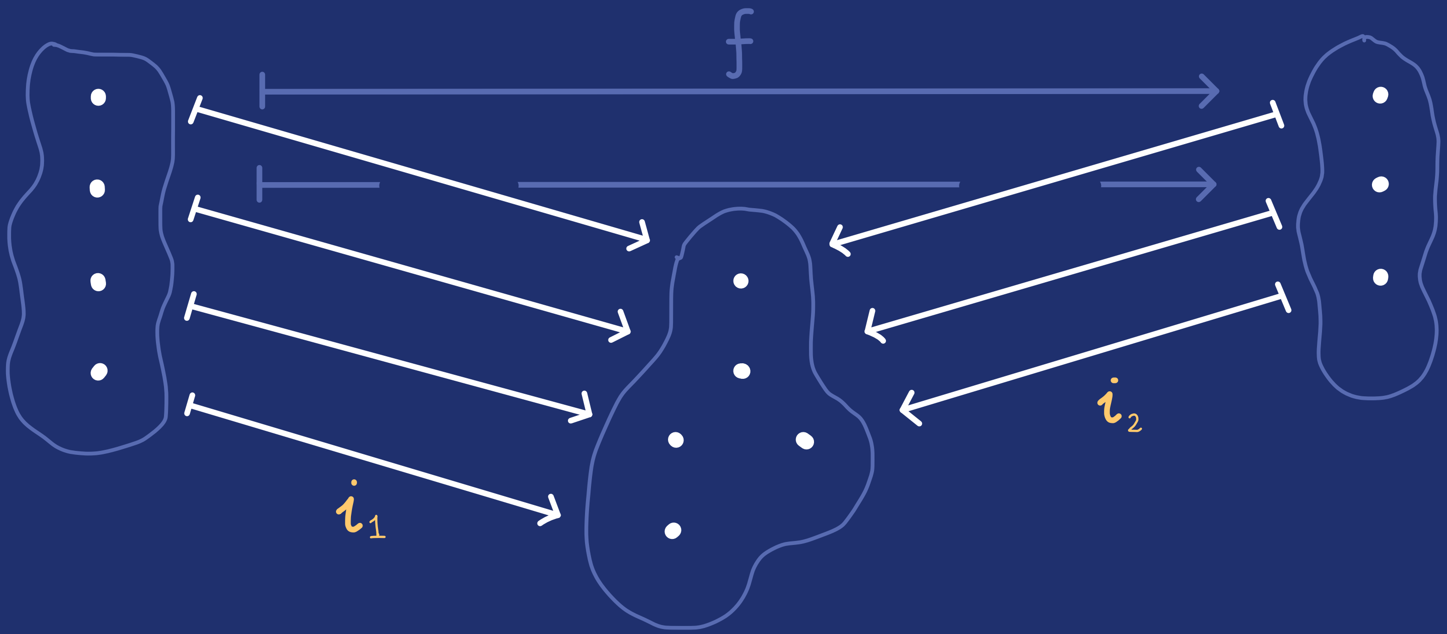

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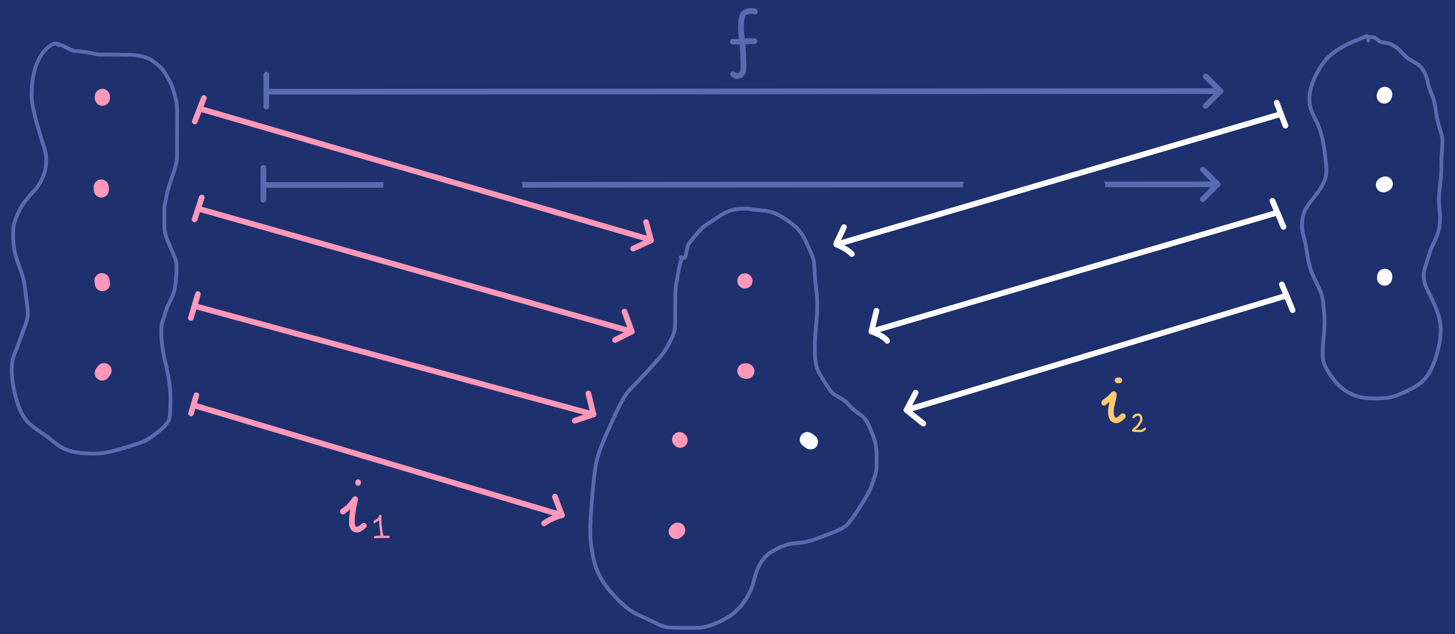

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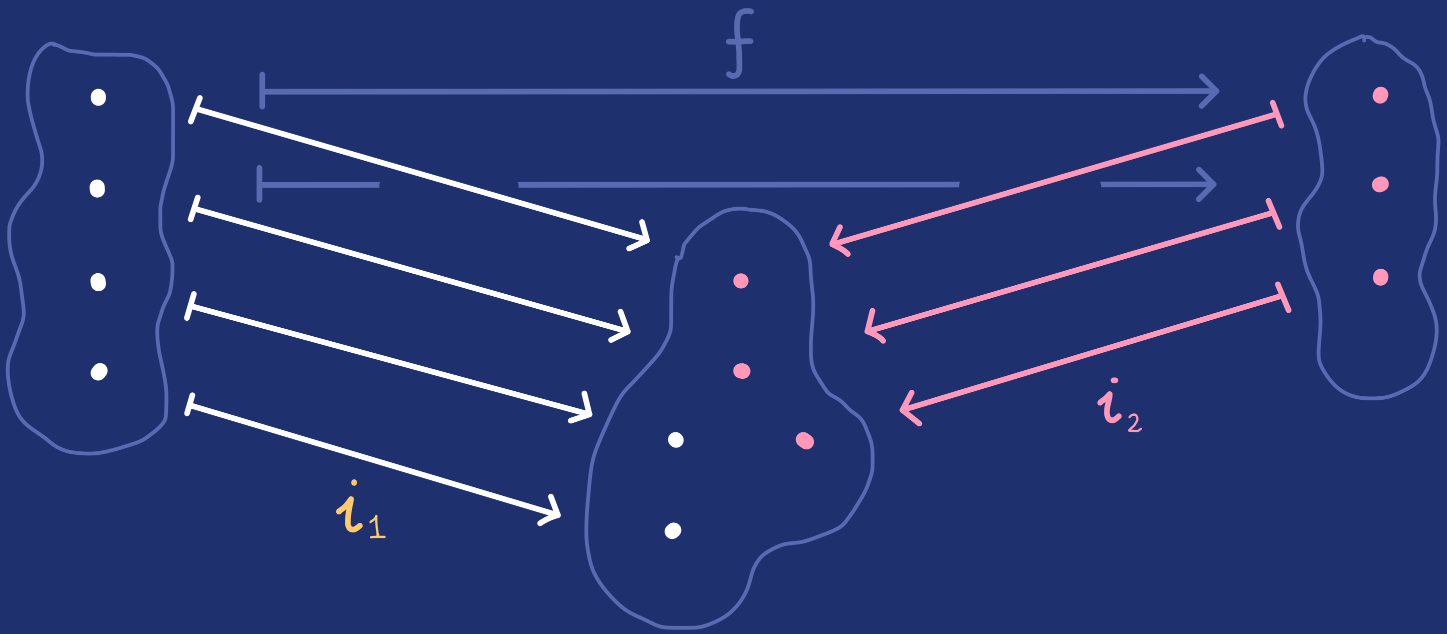

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Finite probability spaces and stochastic maps form a $*$-category FinPS

(with full support)

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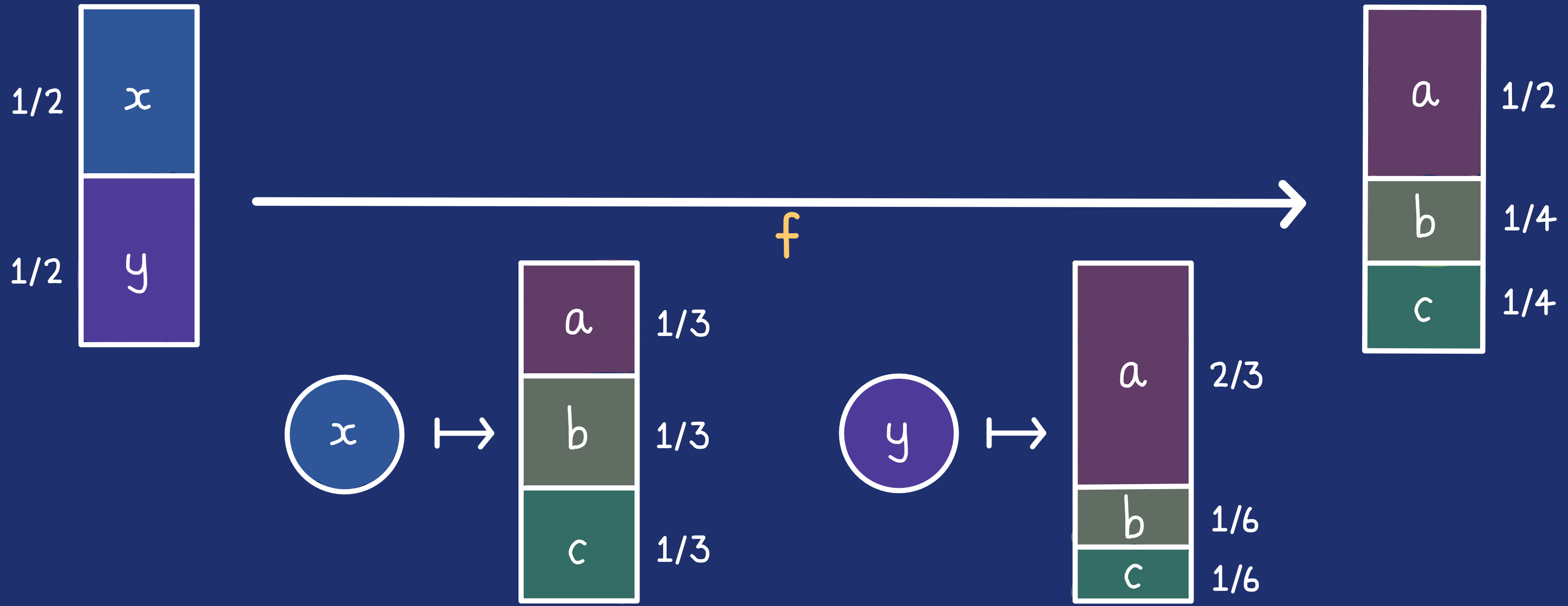

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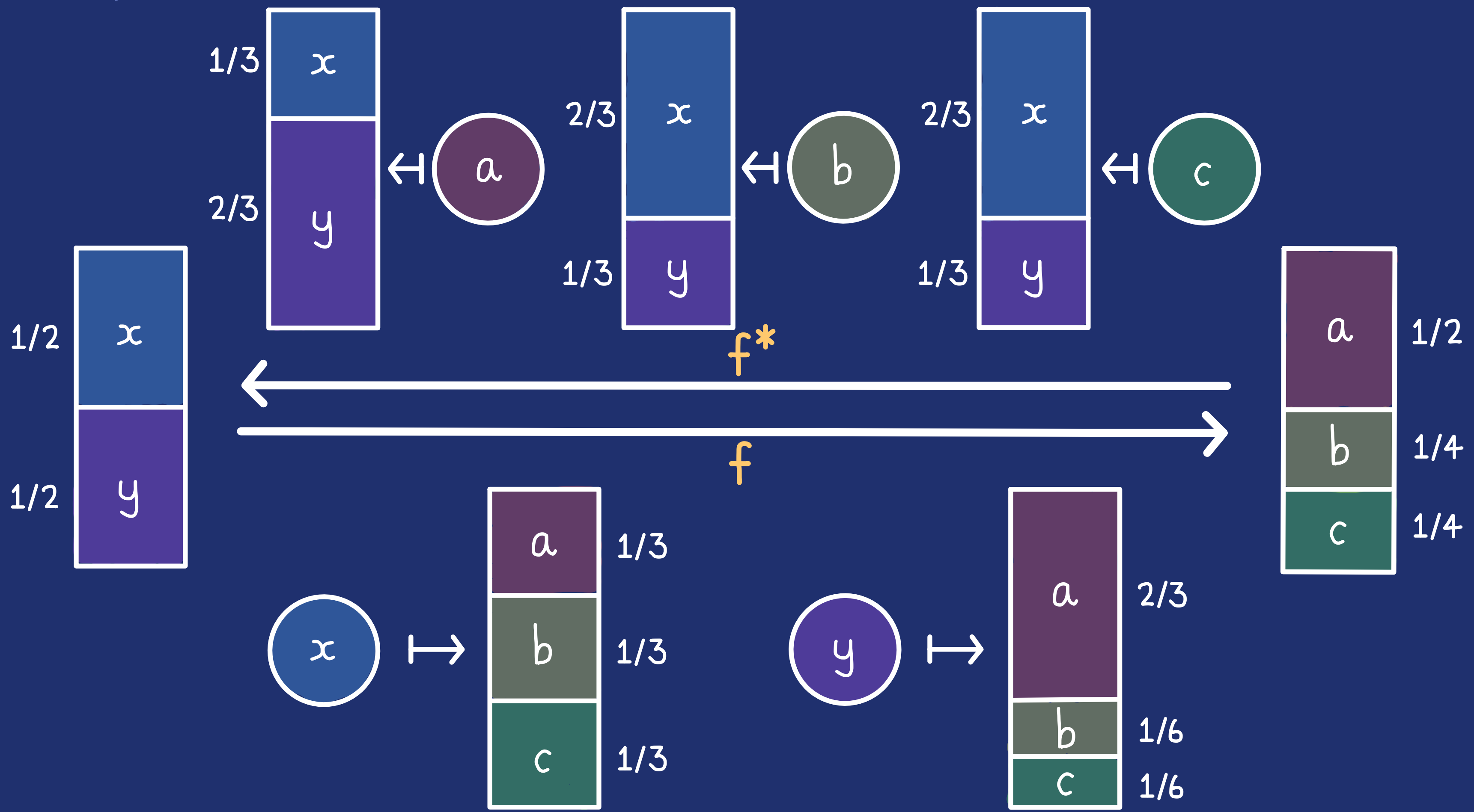

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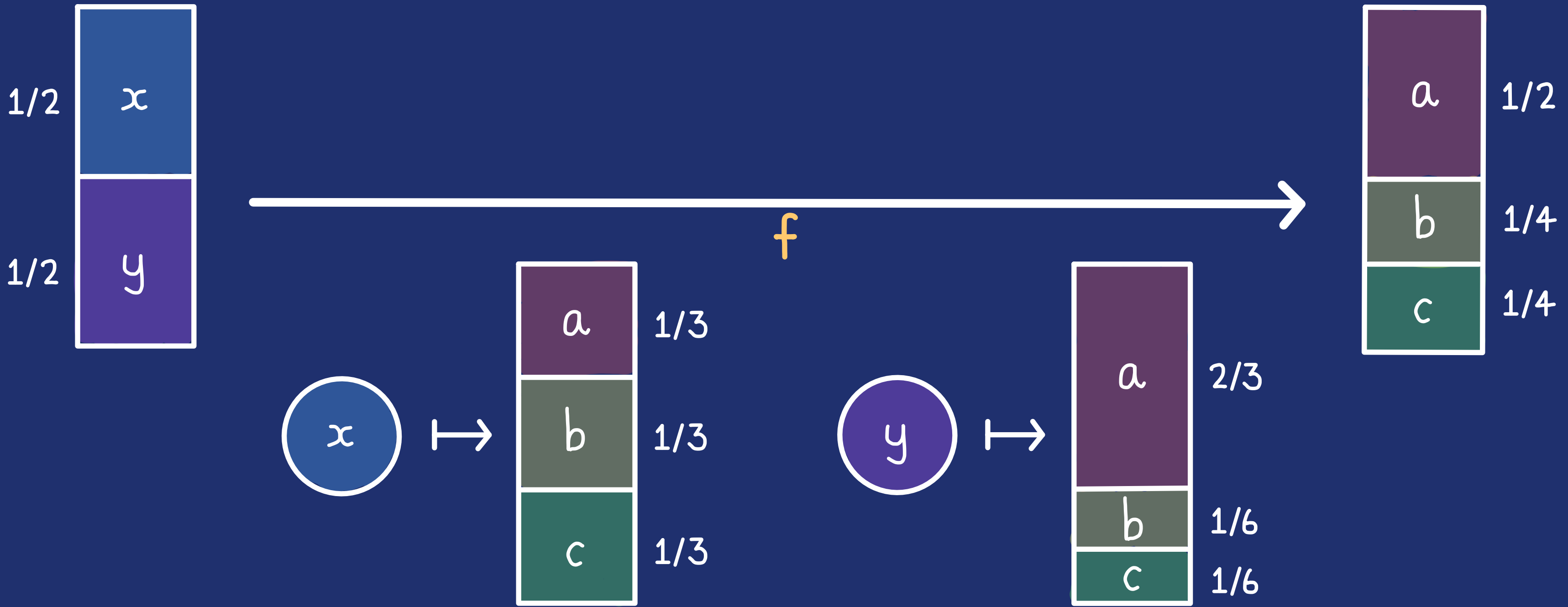
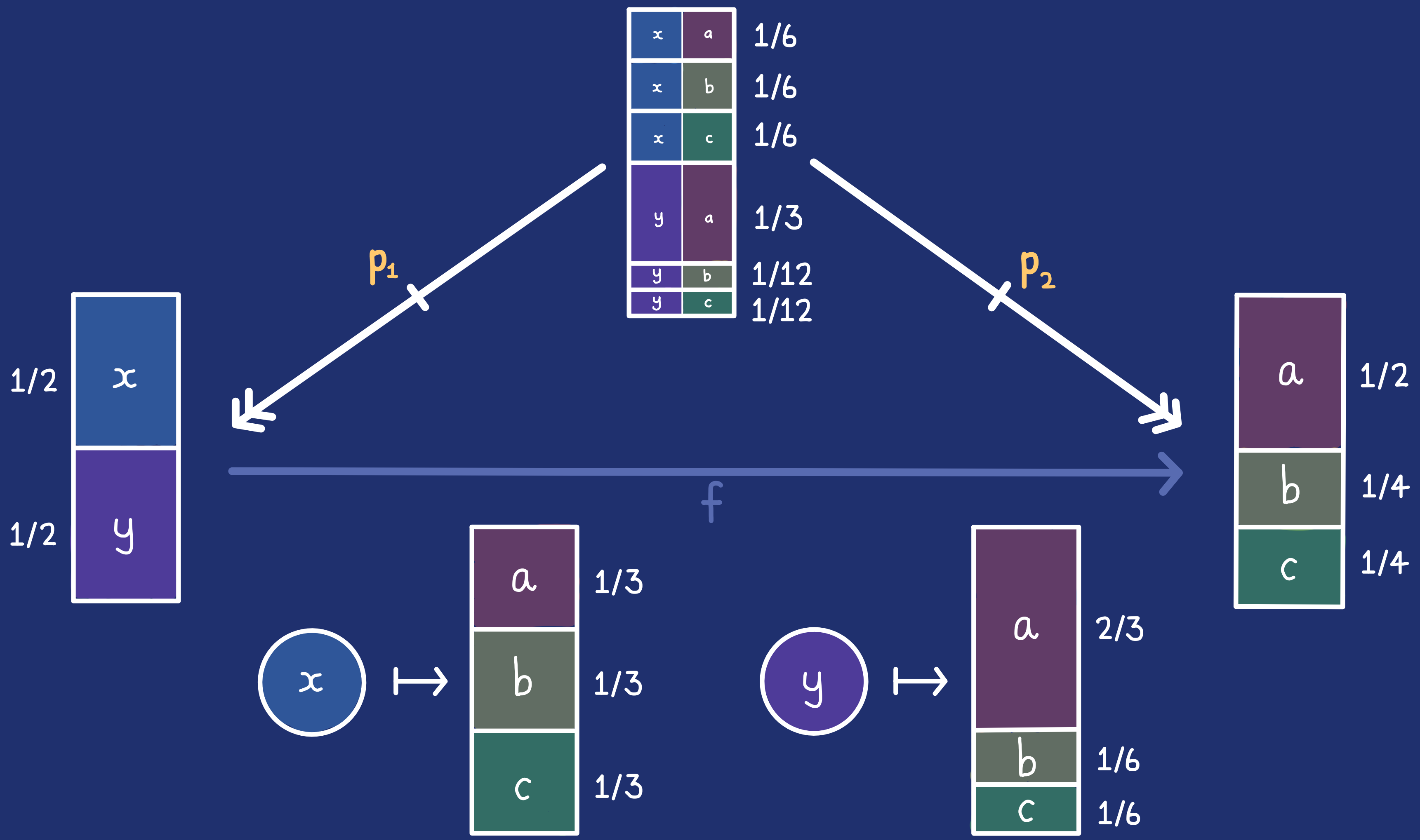


Every morphism in FinPS has a dilator



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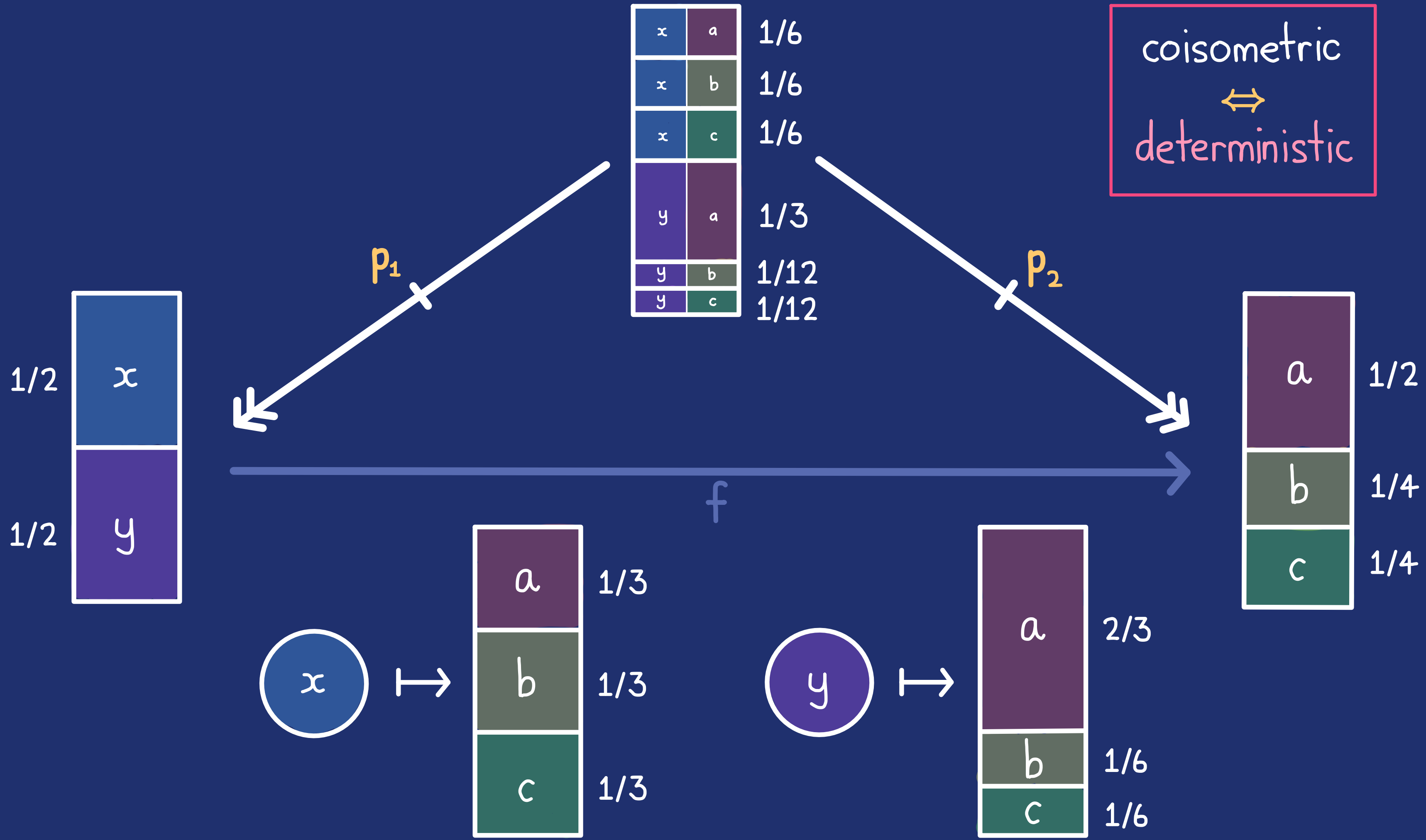


Every morphism in FinPS has a dilator

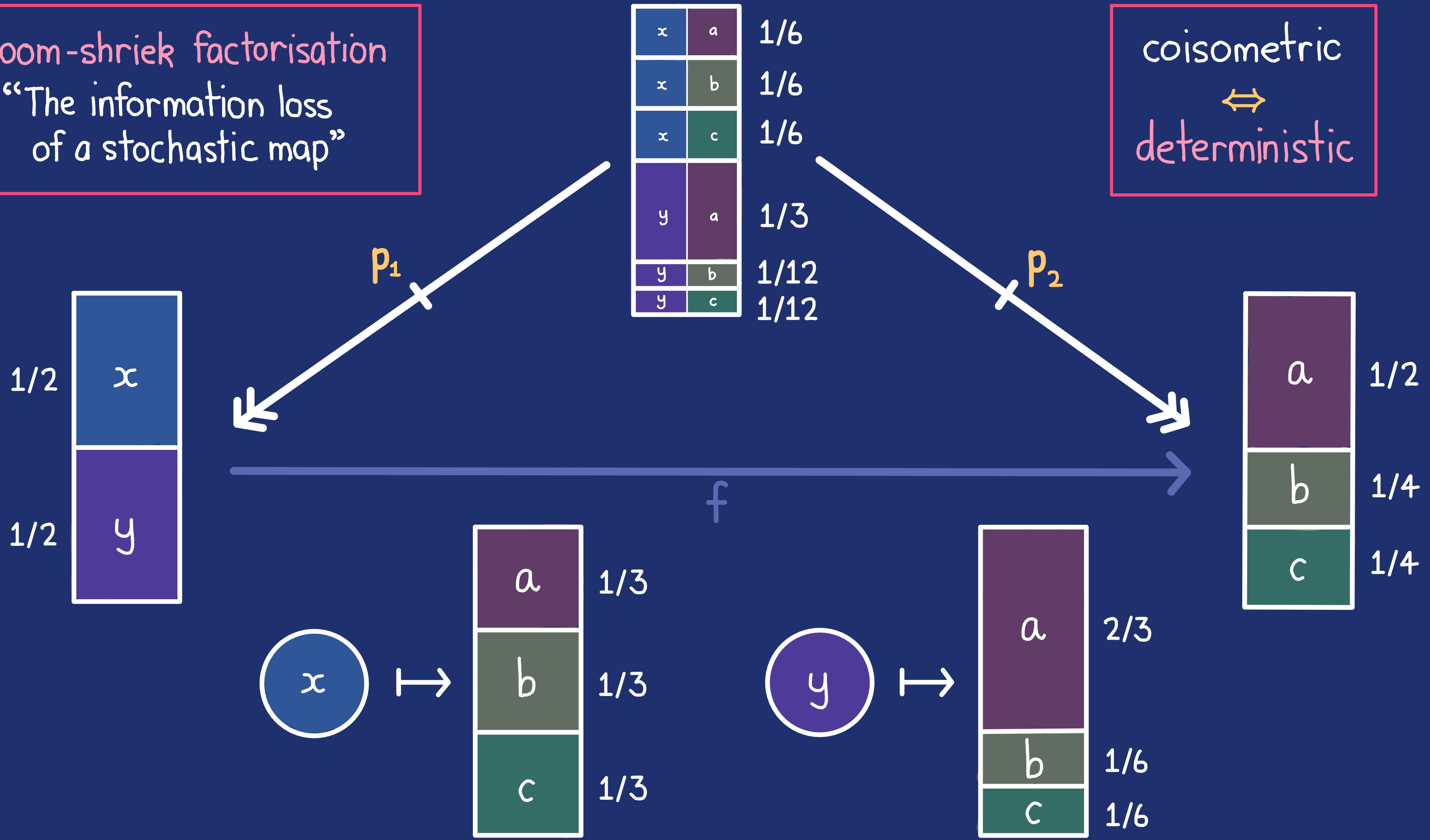
bloom-shriek factorisation

"The information loss of a stochastic map"

coisometric

$\Leftrightarrow$

deterministic



Let C be a $*$-category with

- an enrichment in Ab,
- finite orthoisometric biproducts,

$$i_1^* = p_1 \qquad i_2^* = p_2$$

- isometric kernels.

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$$1 - f^* f = g^* g$$

for some isomorphism $g : X \to Z$.

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$$\begin{tikzcd} & Y \oplus Z & \\ X \arrow[ur, "{\begin{bmatrix} f \\ g \end{bmatrix}}"] & & Y \arrow[ul, "i_1"'] \end{tikzcd}$$

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SUMMARY

- Dilators are a new universal construction in $*$-categories
- They generalise minimal unitary dilations of contractions and the bloom-shriek factorisation of stochastic maps
- Every strict contraction in a nice $*$-category has a dilator

https://mdimeglio.github.io

m.dimeglio@ed.ac.uk

A Dilation of "Minimal Dilations Categorically"

Matthew Di Meglio

Edinburgh Category Theory Seminar
September 2024

Let $\underline{C}$ be a $*$-category in which

(1) a zero object exists

(2) dilators exist

(3) isometric equalisers exist

(4) regular monos are normal

EXAMPLES: $\underline{\mathrm{Hilb}}_{\leq 1}$, $\underline{\mathrm{Rel}}_{\leq 1}$

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PROPOSITION: All cospans have a (jointly epic, isometric) factorisation

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PROPOSITION:
Dilators are jointly epic

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PROPOSITION:
Dilators are jointly epic

LEMMA: $i_1 m_1 = i_2 m_2$

$$\begin{tikzcd}
& X_1 \oplus_A X_2 & \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "m_2 m_1^*"] & & X_2 \arrow[ul, "i_2"'] \\
& A \arrow[ul, "m_1"] \arrow[ur, "m_2"'] &
\end{tikzcd}$$

Let $\underline{C}$ be a $*$-category in which

(1) a zero object exists

(2) dilators exist

(3) isometric equalisers exist

(4) regular monos are normal

EXAMPLES: $\underline{\mathrm{Hilb}}_{\leq 1}$, $\underline{\mathrm{Rel}}_{\leq 1}$

PROPOSITION: All cospans have a (jointly epic, isometric) factorisation

PROPOSITION:

Dilators are jointly epic

LEMMA: $i_1 m_1 = i_2 m_2$

$$
\begin{tikzcd}
& X_1 & & X_2 & \\
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& X_1 \arrow[ur, tail, "i_1"] \arrow[rr, "m_2 m_1^*"] & & X_2 \arrow[ul, tail, "i_2"'] & \\
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$$\boxed{t^* s = f}$$

$$\begin{tikzcd}
X \arrow[d] \arrow[dd, bend right, "f"'] \arrow[dr, bend left, "s"] & \\
X \oplus_f Y & A \\
Y \arrow[u] \arrow[ur, bend right, "t"'] &
\end{tikzcd}$$

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 & X \oplus_s A & \\
X \oplus_f Y & & A \arrow[ul, tail] \arrow[dl, tail] \\
 & Y \oplus_t A & \\
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X \oplus_f Y & & (X \oplus_s A) \oplus_A (Y \oplus_t A) & A \arrow[ul] \arrow[dl] \\
& & Y \oplus_t A \arrow[u] & \\
Y \arrow[uu] \arrow[urr] \arrow[uurrr, bend right, "t"'] & & &
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Y \arrow[uu, tail] \arrow[ur, tail] \arrow[uurr, bend right, "t"'] & &
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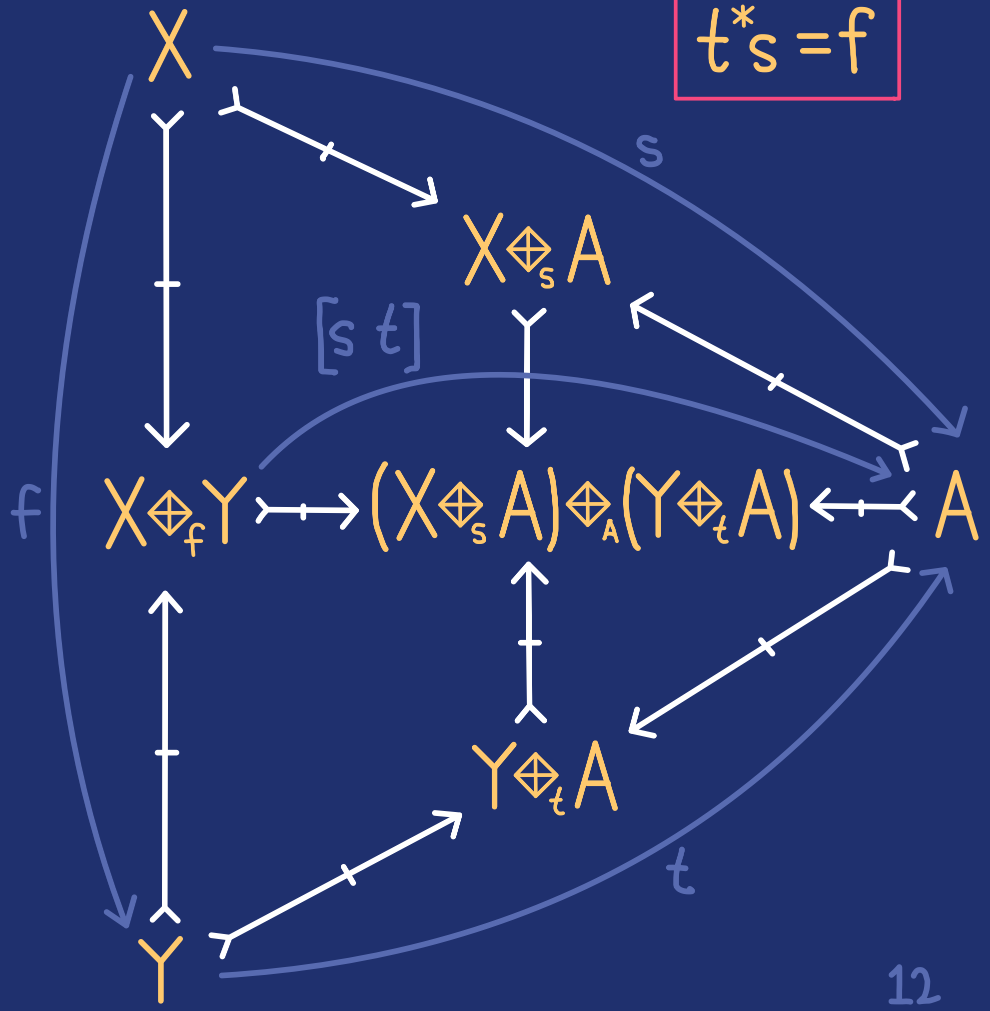

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$$
\begin{tikzcd}
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$$\begin{tikzcd}
X_1 \arrow[r, "i_1"] \arrow[d, "f_1"'] & X_1 \oplus X_2 \arrow[d, "f_1 \oplus f_2"] & X_2 \arrow[l, "i_2"'] \arrow[d, "f_2"] \\
Y_1 \arrow[r, "i_1"'] & Y_1 \oplus Y_2 & Y_2 \arrow[l, "i_2"]
\end{tikzcd}$$

Some Axioms for $\mathbf{Hilb}_{\leq 1}$:

(1) exists a semicartesian monoidal product $(\oplus, 0)$

(2) $X \cong X \oplus 0 \xrightarrow{1 \oplus 0} X \oplus Y$ and $Y \cong 0 \oplus Y \xrightarrow{0 \oplus 1} X \oplus Y$ are jointly epic

(3) If x and y are epic, then $x^* x = y^* y$ iff $y = fx$ for some iso f

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Replacement Axioms:

(A) exists a zero object

(B) every morphism has a dilator

(C) If $x^* x = y^* y$ then $y = fx$ for some f

Discussion Points

- Dilators in ordinary categories?
- Connection to factorisation systems?
- More examples?

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