Polynomial functors and families parametrised by comonoids

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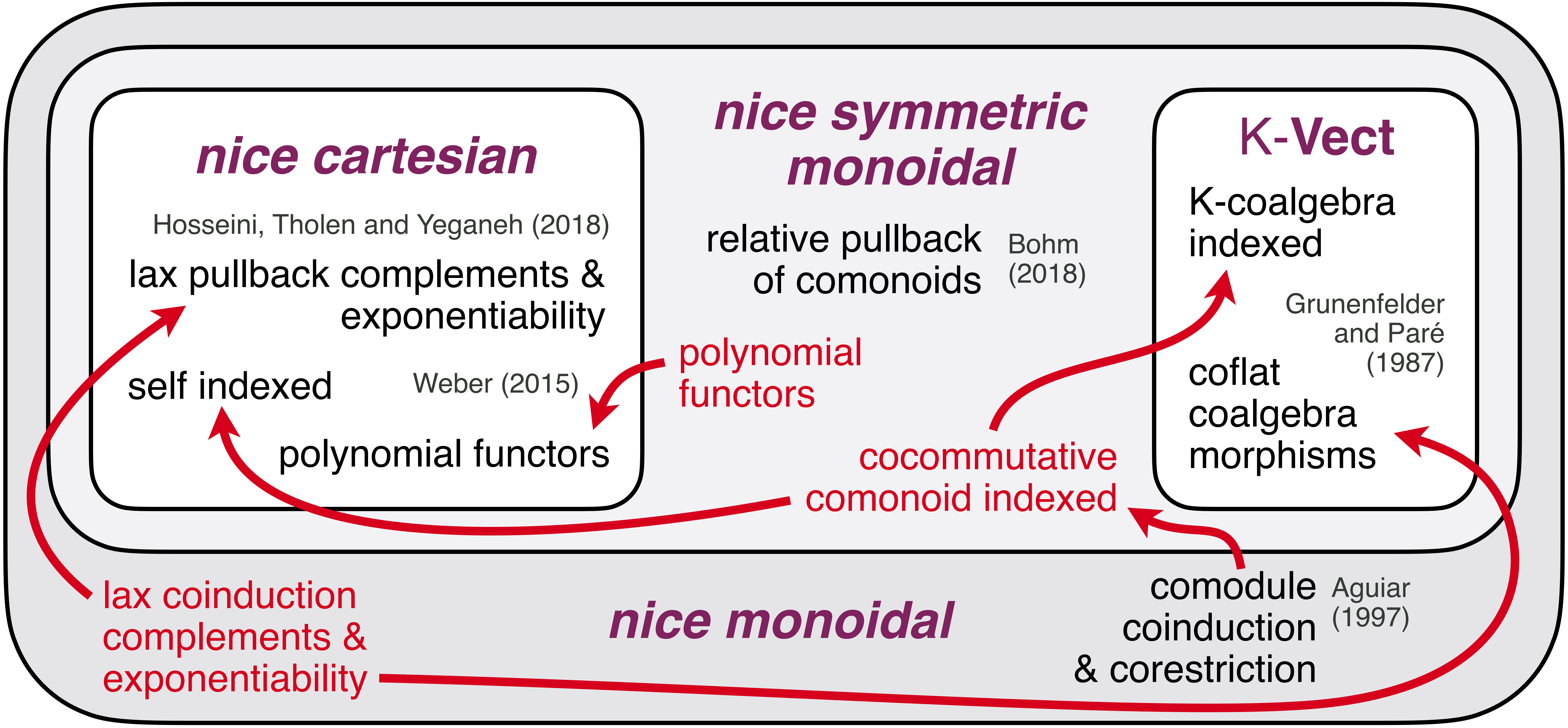

Outline

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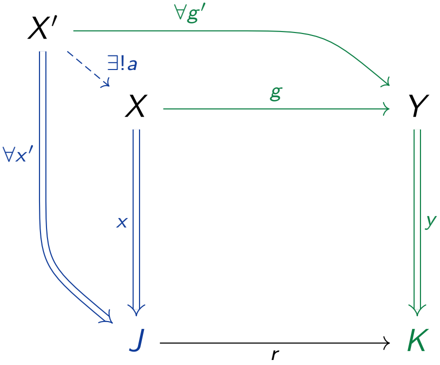
1. Comonoid indexing generalises self indexing
2. Comodule diagrams
3. Polynomial functors in nice monoidal categories

Comonoid indexing generalises self indexing

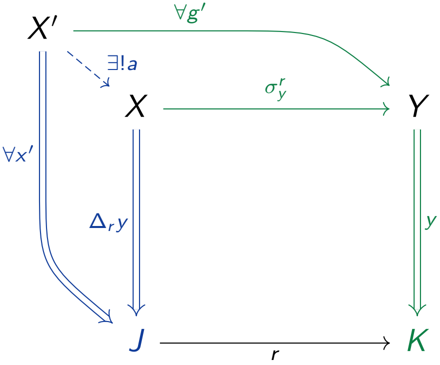
cartesian monoidal category $\mathbf{C}$	$\rightsquigarrow$	symmetric monoidal category $\mathcal{V}$
$\mathbf{C}$	$\rightsquigarrow$	$\mathbf{CComon}_{\mathcal{V}}$
$\mathbf{C}/J$	$\rightsquigarrow$	$\mathbf{Comod}_{\mathcal{V}}J$
composition	$\rightsquigarrow$	corestriction
$\mathbf{C}$ has pullbacks	$\rightsquigarrow$	$\mathcal{V}$ has coreflexive equalisers, and $\otimes$ preserves them in each variable
pullback	$\rightsquigarrow$	coinduction
lax pullback complement (distributivity pullback)	$\rightsquigarrow$	lax coinduction complement

Outline

Comodule diagrams example



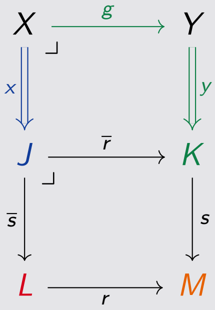
Comodule diagrams example



$$\mathbf{Comod}_{\mathcal{V}} J \underset{\Delta_r}{\overset{\Sigma_r}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Comod}_{\mathcal{V}} K$$

Proposition

The pasting is a generalised pullback square



Outline

Let $\mathbf{C}$ be a category with pullbacks.

A ***polynomial*** in $\mathbf{C}$ is a diagram in $\mathbf{C}$ of shape

$$J \xleftarrow{\;s\;} A \xrightarrow{\;r\;} B \xrightarrow{\;t\;} K$$

where r is exponentiable in $\mathbf{C}$.

The associated ***polynomial functor*** is the composite functor

$$\mathbf{C}/J \xrightarrow{\;\Delta_s\;} \mathbf{C}/A \xrightarrow{\;\Pi_r\;} \mathbf{C}/B \xrightarrow{\;\Sigma_t\;} \mathbf{C}/K$$

In $\mathbf{Set}$, under the isomorphisms $\mathbf{Set}/J \cong \prod_J \mathbf{Set}$,

$$\Sigma_t \Pi_r \Delta_s (X_j)_{j\in J} = \Big(\sum_{b\in t^{-1}k} \prod_{a\in r^{-1}b} X_{sa} \Big)_{k\in K}$$

Polynomial functors in nice monoidal categories

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Let $\mathcal{V}$ be a symmetric monoidal category with coreflexive equalisers, such that $\otimes$ preserves them in each variable.

A ***polynomial*** in $\mathcal{V}$ is a diagram in $\mathbf{CComon}_{\mathcal{V}}$ of shape

$$J \xleftarrow{\ s\ } A \xrightarrow{\ r\ } B \xrightarrow{\ t\ } K$$

where r is exponentiable in $\mathcal{V}$.

The associated ***polynomial functor*** is the composite functor

$$\mathbf{Comod}_{\mathcal{V}}J \xrightarrow{\ \Delta_s\ } \mathbf{Comod}_{\mathcal{V}}A \xrightarrow{\ \Pi_r\ } \mathbf{Comod}_{\mathcal{V}}B \xrightarrow{\ \Sigma_t\ } \mathbf{Comod}_{\mathcal{V}}K$$

Composition of polynomials and polynomial functors

- $\mathbf{CComon}_{\mathcal{V}}$ is a category with pullbacks under the assumptions on $\mathcal{V}$.
- If $U \colon \mathbf{CComon}_{\mathcal{V}} \to \mathcal{V}$ has a right adjoint (i.e. $\mathcal{V}$ has cofree comonoids) then exponentiability in $\mathcal{V}$ implies exponentiability in $\mathbf{CComon}_{\mathcal{V}}$.
- Polynomials in $\mathcal{V}$ compose as polynomials in $\mathbf{CComon}_{\mathcal{V}}$.
- If indexed products which exist distribute over indexed sums, then the mapping from polynomials to polynomial functors is functorial.

Conclusion

nice monoidal

exponentiability,
lax coinduction complements
and comodule diagrams

Ahman and Uustalu (2016)

Garner (2019)

Spivak and Niu (2021)

Poly

bicomodules are category-indexed families of polynomials

comonoid indexed with ?????? sums

????????

polynomial functors

cocommutative comonoid indexed with indexed sums

comonoid indexed with some indexed-ish sums

relative pullback of comonoids

Bohm (2018)

indexed distributivity?

nice symmetric monoidal

more examples?