Polynomial functors and families parametrised by comonoids*

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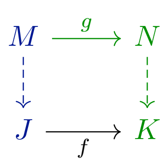
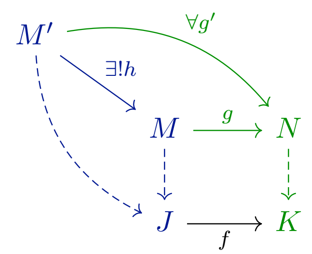
Grunenfelder and Paré [3] explored how, for a field $\mathbb{K}$ and a $\mathbb{K}$-coalgebra J, the J-comodules may be thought of as J-indexed families of $\mathbb{K}$-vector spaces. Aguiar [1] and Bohm [2] studied the comonoids and comodules of a monoidal category $\mathcal{V}$ with equalisers for which the monoidal product bifunctor preserves equalisers in each variable. Much of Grunenfelder and Paré's work generalises to the monoidal categories $\mathcal{V}$ studied by Aguiar and Bohm, with the additional assumptions that the monoidal product is symmetric and that the comonoids under consideration are cocommutative. Part of this generalisation is that $\mathcal{V}$ is an indexed monoidal category, indexed over the category $\mathbf{Comon}_{\mathcal{V}}$ of cocommutative comonoids in $\mathcal{V}$. Given a cocommutative comonoid J, the category of J-indexed families of objects of $\mathcal{V}$ is the category $\mathbf{Comod}_{\mathcal{V}}(J)$ of J-comodules in $\mathcal{V}$. Given a cocommutative comonoid morphism $f\colon J \to K$, the reindexing functor $\Delta_f\colon \mathbf{Comod}_{\mathcal{V}}(K) \to \mathbf{Comod}_{\mathcal{V}}(J)$ is the coinduction functor along f. Specialising to the case where $\mathcal{V}$ is cartesian monoidal (and thus finitely complete), this reduces to the usual self-indexing of $\mathcal{V}$, where the comonoids and comonoid morphisms are just the objects and morphisms of $\mathcal{V}$, the J-comodules for an object J of $\mathcal{V}$ are just the objects of $\mathcal{V}$ over J, and the coinduction functor $\Delta_f\colon \mathbf{Comod}_{\mathcal{V}}(K) \to \mathbf{Comod}_{\mathcal{V}}(J)$ for a morphism $f\colon J \to K$ of $\mathcal{V}$ is just the functor $\Delta_f\colon \mathcal{V}/K \to \mathcal{V}/J$ which pulls back along f.

With a generalised notion of pullback square, there is a new diagramatic approach to proving results about coinduction functors. By the diagram on the left of (1),





(1)

we mean that f is a cocommutative comonoid morphism $J \to K$, M is a J-comodule, N is a K-comodule and g is a K-comodule morphism $\Sigma_g(M) \to N$ where $\Sigma_g(M)$ is the corestriction of M along f. Such a diagram is a *generalised pullback square* if it satisfies the universal property depicted in the diagram on the right of (1). The left Beck Chevalley result for Δ_f amounts to the ability to paste generalised pullback squares with pullback squares in $\mathbf{Comon}_{\mathcal{V}}$; this idea leads to a more direct proof of the left Beck Chevalley result than the one presented by Grunenfelder and Paré.

Weber [5] studied polynomials and polynomial functors in a finitely complete category, introducing the concept of a *distributivity pullback* to describe polynomial composition. Hosseini et al. [4] independently defined the notion of *lax pullback complement* (another name for distributivity pullback). Their calculus of lax pullback complements includes diagramatic versions of several results about the dependent product functors Π_f that Weber needed in order to show that the mapping from polynomials to polynomial functors is functorial. For example, the vertical pasting of lax pullback complement diagrams in their Proposition 4.1 corresponds to $\Pi_{gf} \cong \Pi_g \Pi_f$, and the horizontal pasting of lax pullback complement diagrams in their Proposition 4.2 corresponds to Weber's Proposition 2.2.3.

The notions of exponentiable morphisms, polynomials, polynomial functors, and distributivity pullbacks generalise to $\mathcal{V}$. A *polynomial* in $\mathcal{V}$ is a diagram of shape

$$A \xleftarrow{s} J \xrightarrow{f} K \xrightarrow{t} B$$

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in $\mathbf{Comon}_{\mathcal{V}}$, and the *associated polynomial functor* is the composite functor

$$\mathbf{Comod}_{\mathcal{V}}(A) \xrightarrow{\Delta_s} \mathbf{Comod}_{\mathcal{V}}(J) \xrightarrow{\Pi_f} \mathbf{Comod}_{\mathcal{V}}(K) \xrightarrow{\Sigma_t} \mathbf{Comod}_{\mathcal{V}}(B)$$

where a right adjoint Π_f of Δ_f exists as f is assumed to be generalised exponentiable. Many of the results that Weber uses to prove that the mapping from polynomials to polynomial functors is functorial generalise to this setting.

References

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