The category of lenses is regular-ish

Matthew Di Meglio

MACQUARIE University

Australian Category Seminar

Outline

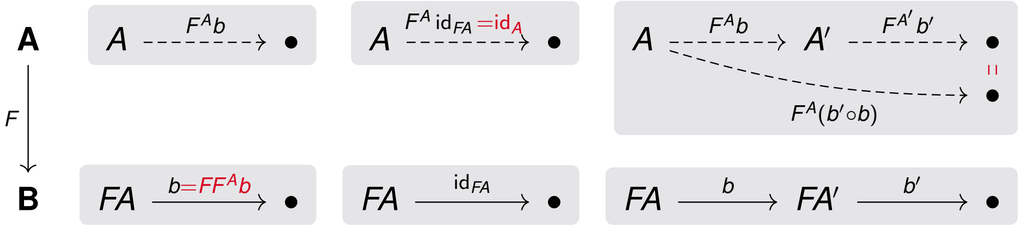
1. The category **Lens**
2. **Lens** is regular-ish
3. Monos, epis and images in **Lens**
4. All epis in **Lens** are proxy effective

What is a lens?

A ***lens*** $F\colon \mathbf{A} \to \mathbf{B}$ consists of

- a ***get functor*** $F\colon \mathbf{A} \to \mathbf{B}$, and
- a ***put function*** $F^A\colon \mathbf{B}(FA, \bullet) \to \mathbf{A}(A, \bullet)$ for each $A \in |\mathbf{A}|$,

such that



(PutGet) (PutId) (PutPut)

The category of lenses

- The composite $G \circ F$ of lenses $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ is given by

$$
\begin{aligned}
(G \circ F)A &= GFA \\
(G \circ F)a &= GFa \\
(G \circ F)^{A}c &= F^{A}G^{FA}c
\end{aligned}
$$

- **Lens** is the category of small categories and lenses
- $U\colon \mathbf{Lens} \to \mathbf{Cat}$ is the functor that sends each lens to its get functor

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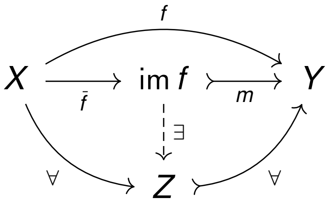
Regular categories

A category is ***regular*** if

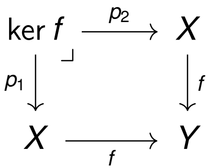
- it has all finite limits,
- it has image factorisations



  of each $f\colon X \to Y$, and
- image factorisations are pullback stable.

Equivalently, a category is ***regular*** if

- it has all finite limits,
- the kernel pair of each $f\colon X \to Y$



  has a coequaliser, and
- regular epis are pullback stable.

Properties of regular categories

For a morphism e in a category $\mathbf{C}$

$$
\begin{array}{ccccccc}
e \text{ effective epi} & \underset{\Longleftarrow}{\Longrightarrow} & e \text{ regular epi} & \underset{\Longleftarrow}{\Longrightarrow} & e \text{ strong epi} & \underset{\Longleftarrow}{\Longrightarrow} & e \text{ extremal epi} \\
 & e \text{ has a kernel pair} & & \mathbf{C} \text{ is regular} & & \mathbf{C} \text{ has pullbacks of monos} &
\end{array}
$$

Proposition

In a regular category

- *regular epis and monos form an orthogonal factorisation system*
- *effective epis, regular epis, strong epis and extremal epis coincide*

Internal equivalence relations

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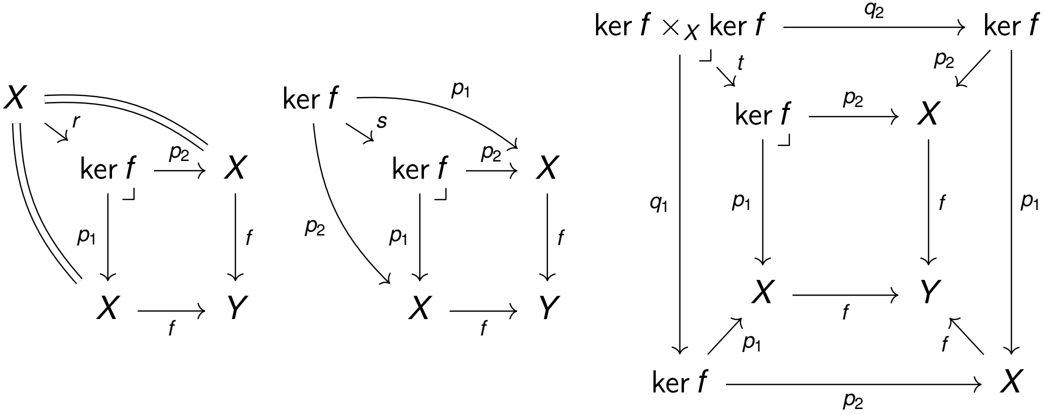
- kernel pairs are internal equivalence relations



- a kernel pair's coequaliser is like the object of its equivalence classes

Limits in **Lens**

- equalisers
- a terminal object (the terminal object **1** of **Cat**)
- not all products (e.g. not **2** $\times$ **2**)
- canonical ***proxy pullbacks***
  (these have some similar properties to pullbacks)
- canonical ***proxy products***
  (the proxy pullback of the unique cospan over the terminal object)
- canonical ***proxy kernel pairs***
  (the proxy pullback of a morphism along itself)

Proxy pullbacks in **Lens**

In **Lens**, a span $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ is a ***proxy pullback*** of $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$ if

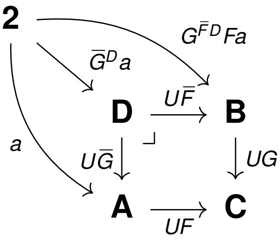
- $\begin{array}{ccc} \mathbf{D} & \xrightarrow{\overline{F}} & \mathbf{B} \\ {\scriptstyle \overline{G}}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array}$ is a commuting square in **Lens**,
- $\begin{array}{ccc} \mathbf{D} & \xrightarrow{U\overline{F}} & \mathbf{B} \\ {\scriptstyle U\overline{G}}\downarrow & \lrcorner & \downarrow{\scriptstyle UG} \\ \mathbf{A} & \xrightarrow[UF]{} & \mathbf{C} \end{array}$ is a pullback square in **Cat**, and
- for all $D \in |\mathbf{D}|$, all $a \in \mathbf{A}(\overline{G}D, \bullet)$ and all $b \in \mathbf{B}(\overline{F}D, \bullet)$,

$$\overline{F}\,\overline{G}^D a = G^{\overline{F}D} Fa \qquad \text{and} \qquad \overline{G}\,\overline{F}^D b = F^{\overline{G}D} Gb.$$

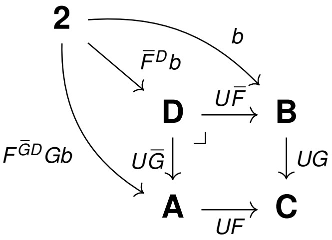
Proxy pullback properties

$$\overline{F}\,\overline{G}^D a = G^{\overline{F}D} F a \qquad \overline{G}\,\overline{F}^D b = F^{\overline{G}D} G b$$

$\iff$



$\iff$



Proposition

Each pullback of the get functors of a lens cospan lifts uniquely to a proxy pullback of the cospan

Proxy pullbacks are pullback-ish

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Proposition

Proxy pullbacks are unique up to unique isomorphism

Proposition (proxy pullback stability)

The following classes of morphisms in **Lens** *are proxy pullback stable:*

- *identity morphisms*
- *isomorphisms*
- *discrete opfibrations*
- *split opfibrations*

Proxy pullbacks are pullback-ish

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Proposition (proxy pullback pasting)

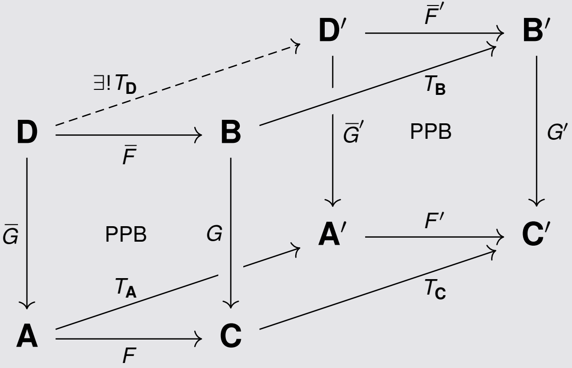
$$\begin{array}{ccccc} \mathbf{D} & \xrightarrow{\bar{F}} & \mathbf{B} & \longrightarrow & \mathbf{B}' \\ {\scriptstyle \bar{G}}\downarrow & & {\scriptstyle G}\downarrow & \text{PPB} & \downarrow \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} & \longrightarrow & \mathbf{C}' \end{array}$$

$$\textit{left square is PPB} \implies \textit{outer rectangle is PPB}$$

$$\textit{left square is PPB} \impliedby \begin{cases} \textit{outer rectangle is PPB} \\ + \\ \bar{G}\bar{F}^{D} b = F^{\bar{G}D} G b \qquad \forall D \in |\mathbf{D}|, b \in \mathbf{B}(\bar{F}D, \bullet) \end{cases}$$

Proxy pullbacks are pullback-ish

Proposition (constrained naturality)

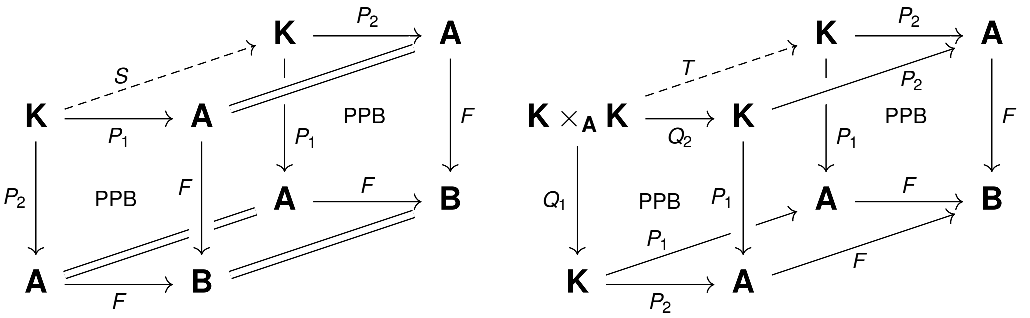


$$F\,T_{\mathbf{A}}{}^{A}a' = T_{\mathbf{C}}{}^{FA}F'a'$$
$$G\,T_{\mathbf{B}}{}^{B}b' = T_{\mathbf{C}}{}^{GB}G'b'$$

$$\bar{F}\,T_{\mathbf{D}}{}^{D}d' = T_{\mathbf{B}}{}^{\bar{F}D}\bar{F}'d'$$
$$\bar{G}\,T_{\mathbf{D}}{}^{D}d' = T_{\mathbf{A}}{}^{\bar{G}D}\bar{G}'d'$$

Proxy pullbacks are pullback-ish

- All proxy kernel pairs have symmetry and transitivity lenses



- The proxy kernel pair of a lens F has a reflexivity lens if and only if F is a discrete opfibration; in this case, it is a real kernel pair

Lens is regular-ish

- proxy effective, regular, strong and extremal epis coincide
  (all epis are proxy effective)
- (regular) epis and monos form an orthogonal factorisation system
- image factorisations are proxy pullback stable
- (regular) epis are proxy pullback stable
- the corestriction of a lens coequalises the lens' proxy kernel pair

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Monos in **Lens**

Proposition

U preserves and reflects monos

Reflection was proved by Chollet et al.

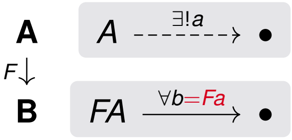
Proof of preservation.

Let $M \colon \mathbf{A} \to \mathbf{B}$ be a monic lens and P_1, P_2 its proxy kernel pair

- $P_1 = P_2$ as M is monic and $M \circ P_1 = M \circ P_2$
- UM is monic as UP_1, UP_2 is its kernel pair and $UP_1 = UP_2$ $\square$

Monic lenses are cosieves

A functor $F \colon \mathbf{A} \to \mathbf{B}$ such that



is called a ***discrete opfibration***.

A ***cosieve*** is an injective-on-objects discrete opfibration.

Proposition

U induces a bijective correspondence between monic lenses and cosieves

Proof.

If $F \colon \mathbf{A} \to \mathbf{B}$ is a monic lens, UF is

- monic by preservation,
- injective on objects and morphisms as it is monic,
- a discrete opfibration as F is an injective-on-morphisms lens.

If $\overline{F} \colon \mathbf{A} \to \mathbf{B}$ is a cosieve, then

- there is a unique lens $F \colon \mathbf{A} \to \mathbf{B}$ such that $UF = \overline{F}$, and
- F is monic by reflection. □

Other properties of monic lenses

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Proposition

Proxy pullbacks along monos are pullbacks

Corollary

Monic lenses are proxy pullback stable

Proposition

A lens $F\colon \mathbf{A} \to \mathbf{B}$ is monic if and only if $\mathrm{id}_{\mathbf{A}}, \mathrm{id}_{\mathbf{A}}$ is a proxy kernel pair of F

Proposition

A lens with proxy kernel pair P_1, P_2 is monic if and only if $P_1 = P_2$; if so, P_1 is iso

Image factorisations in **Lens**

- Cosieves are out-degree-zero subcategory inclusions
- The ***image*** of a lens $F\colon \mathbf{A} \to \mathbf{B}$ is the out-degree-zero subcategory $\mathbf{Im}\,F$ of $\mathbf{B}$ formed by the images of the object and morphism maps of F
- Every lens $F\colon \mathbf{A} \to \mathbf{B}$ has a factorisation

$$\mathbf{A} \xrightarrow{\;E\;} \mathbf{Im}\,F \overset{M}{\rightarrowtail} \mathbf{B}, \qquad F = M \circ E$$

where M is monic and E is surjective on objects and morphisms

Epis in **Lens** are nicer than epis in **Cat**

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Remark

In **Cat**

$$\text{epic} \implies \text{surjective on objects}$$

$$\text{epic} \not\Longrightarrow \text{surjective on morphisms}$$

$$\text{epic} \Longleftarrow \begin{cases} \text{surjective on objects} \\ + \\ \text{surjective on morphisms} \end{cases}$$

Proposition

In **Lens**

$$\textit{epic} \iff \textit{surjective on objects}$$

$$\iff \textit{surjective on morphisms}$$

Cokernels of get functors lift uniquely

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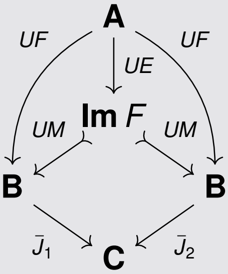
Proposition

Let $F\colon \mathbf{A} \to \mathbf{B}$ be a lens, and $\bar{J}_1, \bar{J}_2\colon \mathbf{B} \to \mathbf{C}$ the cokernel pair of UF. Then there are unique lenses J_1 and J_2 above $\bar{J}_1$ and $\bar{J}_2$, and $J_1 \circ F = J_2 \circ F$.

Proof.



- Let $F = M \circ E$ be the image factorisation of F
- $\bar{J}_1 \circ UM = \bar{J}_2 \circ UM$ as $\bar{J}_1 \circ UF = \bar{J}_2 \circ UF$ and UE is epic
- $\bar{J}_1, \bar{J}_2$ is also the cokernel pair of UM
- $\bar{J}_1$ and $\bar{J}_2$ are cosieves as UM is a cosieve and cosieves are pushout stable
- discrete opfibrations are uniquely lenses □

U preserves epis

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Proposition

U preserves and reflects epis

Reflection was proved by Chollet et al.

Proof of preservation.

Let $E\colon \mathbf{A} \to \mathbf{B}$ be an epic lens and J_1, J_2 the unique lenses above the cokernel pair of UE

- $J_1 = J_2$ as E is epic and $J_1 \circ E = J_2 \circ E$
- UE is epic as UJ_1, UJ_2 is its cokernel pair and $UJ_1 = UJ_2$ $\square$

Epic lenses are proxy pullback stable

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Proposition

Epic lenses are proxy pullback stable

Proof.

$$
\begin{array}{ccc}
\mathbf{D} & \xrightarrow{\overline{F}} & \mathbf{B} \\
{\scriptstyle \overline{E}}\downarrow & \text{PPB} & \downarrow{\scriptstyle E} \\
\mathbf{A} & \xrightarrow[F]{} & \mathbf{C}
\end{array}
$$

Suppose that E is epic. For each $A \in |\mathbf{A}|$

- there is a $B \in |\mathbf{B}|$ with $EB = FA$ as E is surjective on objects, and
- there is a unique $D \in |\mathbf{D}|$ with $\overline{E}D = A$ and $\overline{F}D = B$ as the square of get functors is a pullback in **Cat**.

So $\overline{E}$ is surjective on objects, and thus epic. $\square$

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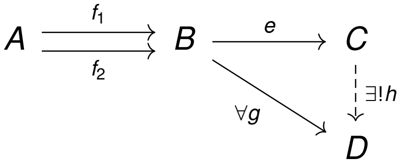
Coequalisers in **Lens**

- *e* ***coequalises*** f_1 and f_2 if it is their universal ***cofork***



- **Cat** has all coequalisers, but they aren't usually nice to describe
- **Lens** doesn't have all coequalisers, nor does *U* preserve/reflect them
- **Lens** does have some coequalisers, some of which are reflected by *U*

Coequaliser non-existence and non-preservation

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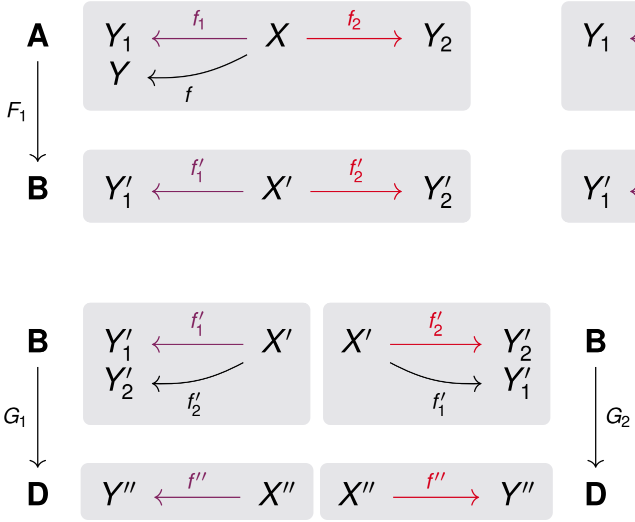

- G_1 and G_2 are the only lenses above the coequaliser of UF_1 and UF_2
- No lens $H \colon \mathbf{D} \to \mathbf{D}$ with $G_1 = H \circ G_2$

Every epic lens is proxy effective

Theorem

Every epic lens coequalises its proxy kernel pair in **Lens**

Proof idea.

- If the comparison lens to a cofork exists, surjectivity gives equations which determine it
- From the coforking property, these equations give a well defined comparison lens

Corollary

All epic lenses are regular, strong and extremal

Corollary

The lenses left orthogonal to all monic lenses are the epic lenses

Corollary

Monic epic lenses are isomorphisms

Every monic lens is effective

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Theorem

U creates pushouts of monic lenses with discrete opfibrations

Corollary

U creates cokernel pairs; the cokernel pair of a lens is the cokernel pair of the inclusion of its image in its target

Proposition

Every monic lens equalises its cokernel pair in **Lens**

Conclusion

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- **Lens** is not complete, but it does have equalisers and proxies for pullbacks, products and kernel pairs
- Epis and monos in **Lens** have simple characterisations
- **Lens** is regular-ish
  - proxy effective, regular, strong and extremal epis coincide (all epis are proxy effective)
  - (regular) epis and monos form a proxy-pullback-stable orthogonal factorisation system

Future work

- General theory of proxy pullbacks and regular-ish categories?
- Symmetric lenses as relations in **Lens**?