Proxy pullbacks in the category of lenses

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Australian Category Seminar

Goal

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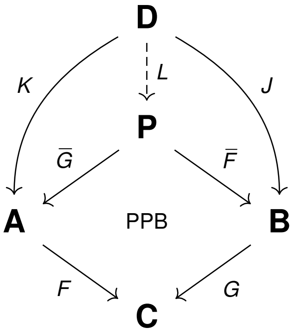
When is there a lens L such that $K = \overline{G} \circ L$ and $J = \overline{F} \circ L$?



L exists $\Longrightarrow$ (K, J) ***independent*** and ***compatible*** with (F, G)

L exists $\Longleftarrow$ (with $(\overline{G}, \overline{F})$ ***sync minimal***) (K, J) ***independent*** and ***compatible*** with (F, G)

$(\overline{G}, \overline{F})$ ***sync minimal*** $\Longleftrightarrow$ L exists for all (K, J) ***independent*** and ***compatible*** with (F, G)

Outline

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1. Proxy pullbacks
2. Compatibility and independence
3. Sync minimality
4. Special cases

Cofunctors

A ***cofunctor*** $F\colon \mathbf{A} \to \mathbf{B}$ consists of

- a function $F\colon |\mathbf{A}| \to |\mathbf{B}|$, called the ***object function***, and
- for each $A \in |\mathbf{A}|$, a function

$$F^A\colon \mathbf{B}(FA, *) \to \mathbf{A}(A, *),$$

  called a ***put function***,

such that the equations

$$F \operatorname{tgt} F^A b = \operatorname{tgt} b \quad \text{(PutTgt)} \qquad F^A \operatorname{id}_{FA} = \operatorname{id}_A \quad \text{(PutId)} \qquad F^A(b' \circ b) = F^{A'} b' \circ F^A b \quad \text{(PutPut)}$$

hold whenever they are defined.

Mixed diagrams and compatible mixed squares

- Mixed diagrams involve $\mathbf{A} \longrightarrow\!\!\!\!\!\!\rightarrow \mathbf{B}$ functors, $\mathbf{A} \longleftarrow\!\!\!\!\!\!\rightarrow \mathbf{B}$ cofunctors, $\mathbf{A} \longrightarrow \mathbf{B}$ lenses

- $$\begin{array}{ccc} \mathbf{D} & \xrightarrow{J} & \mathbf{B} \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array}$$ is a ***compatible mixed square*** if the equations

$$GJD = FKD \qquad \text{and} \qquad JK^{D}a = G^{JD}Fa$$

  hold whenever they are defined

- Categories, functors, cofunctors and compatible mixed squares form a double category

Lenses and discrete opfibrations

- A ***lens*** $F\colon \mathbf{A} \to \mathbf{B}$ is a compatible mixed square

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathcal{G}F} & \mathbf{B} \\ {\scriptstyle \mathcal{P}F}\updownarrow & & \| \\ \mathbf{B} & = & \mathbf{B} \end{array}$$

- $\mathcal{G}F$ is the ***get functor*** of F and $\mathcal{P}F$ is the ***put cofunctor*** of F
- A lens $F\colon \mathbf{A} \to \mathbf{B}$ is a ***discrete opfibration*** if

$$\begin{array}{ccc} \mathbf{A} & = & \mathbf{A} \\ \| & & \updownarrow{\scriptstyle \mathcal{P}F} \\ \mathbf{A} & \xrightarrow[\mathcal{G}F]{} & \mathbf{B} \end{array}$$

is also a compatible mixed square

Compatible lens squares

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{J} & \mathbf{B} \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array}$$ is a ***compatible lens square*** if it commutes and

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{\mathcal{G}J} & \mathbf{B} \\ {\scriptstyle \mathcal{P}K}\downarrow & & \downarrow{\scriptstyle \mathcal{P}G} \\ \mathbf{A} & \xrightarrow[\mathcal{G}F]{} & \mathbf{C} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{D} & \xrightarrow{\mathcal{P}J} & \mathbf{B} \\ {\scriptstyle \mathcal{G}K}\downarrow & & \downarrow{\scriptstyle \mathcal{G}G} \\ \mathbf{A} & \xrightarrow[\mathcal{P}F]{} & \mathbf{C} \end{array}$$

are compatible mixed squares.

Proxy pullbacks

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- A ***proxy pullback square*** is a compatible lens square whose get functors form a pullback square in $\mathcal{C}at$
- For each lens cospan, there is a unique proxy pullback of the cospan above each pullback of the get functors of the cospan
- Proxy pullbacks are unique up to unique isomorphism of lens spans
- Proxy pullbacks are sometimes but not always real pullbacks

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Necessity of compatibility

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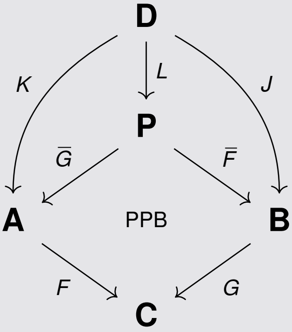
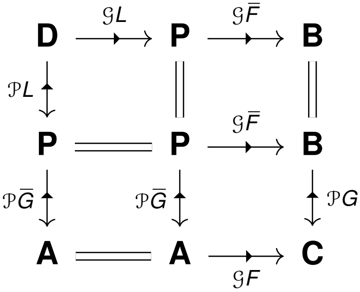
Proposition

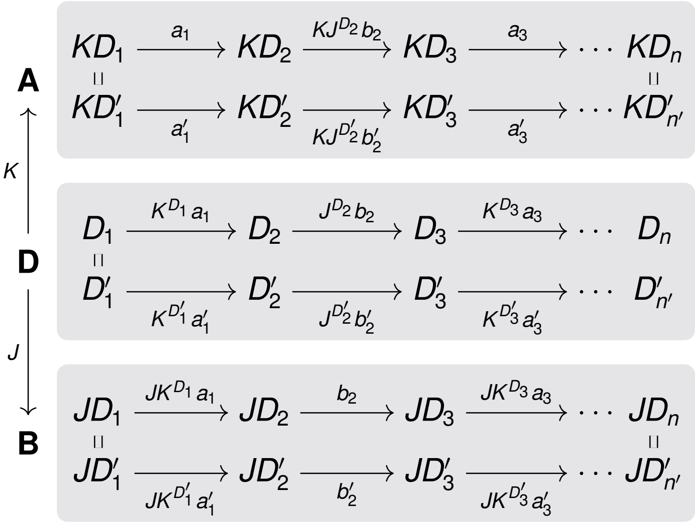
(K, J) is compatible with (F, G)





Independence

A lens span $\mathbf{A} \xleftarrow{K} \mathbf{D} \xrightarrow{J} \mathbf{B}$ is ***independent*** if, for all morphisms d and d' in $\mathbf{D}$ with the same source that are composites of lifts along K and J, whenever $Kd = Kd'$ and $Jd = Jd'$ also $d = d'$.



Necessity of independence

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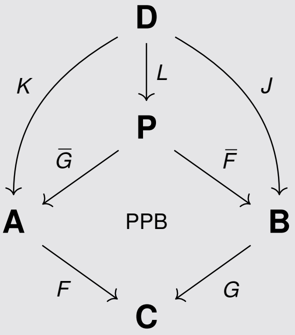
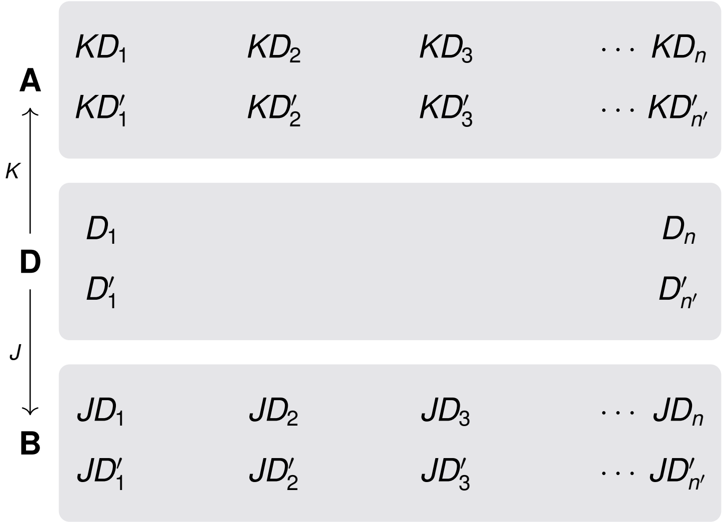
Proposition

(K, J) *is independent*





Necessity of independence

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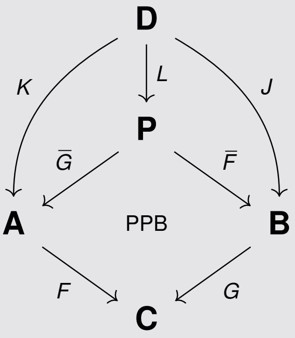
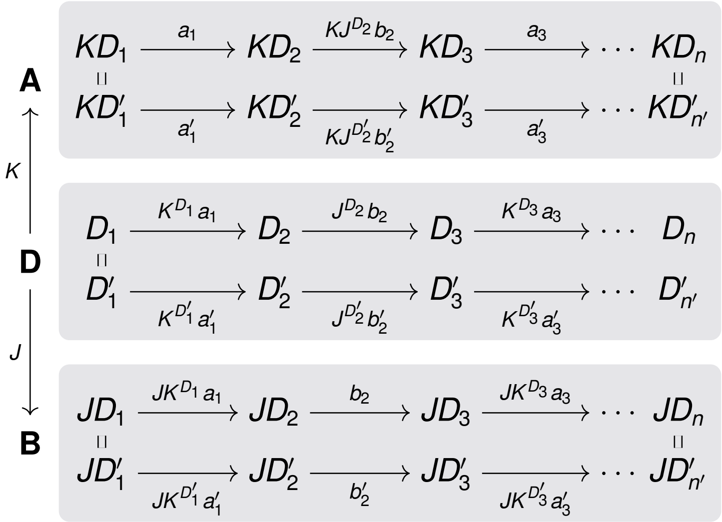
(K, J) is independent





Necessity of independence

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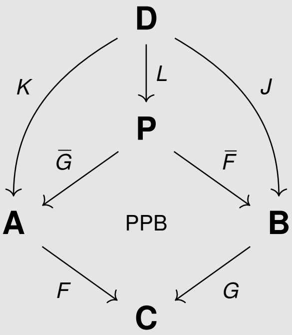

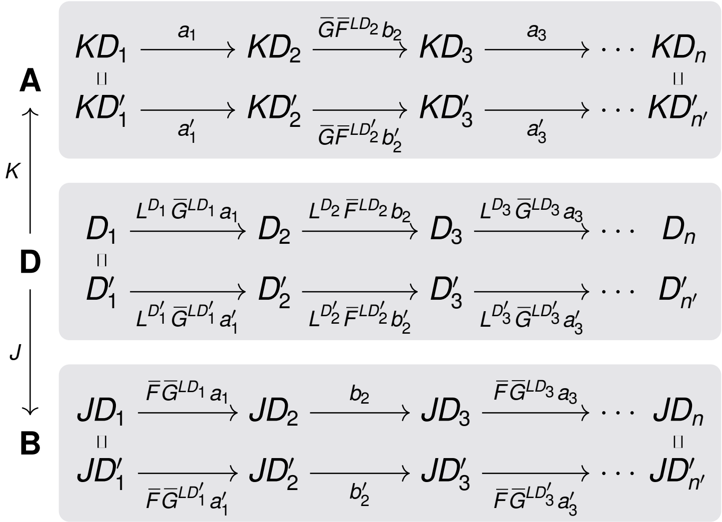

Necessity of independence

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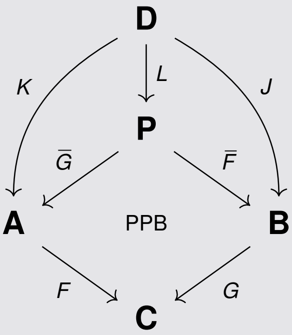
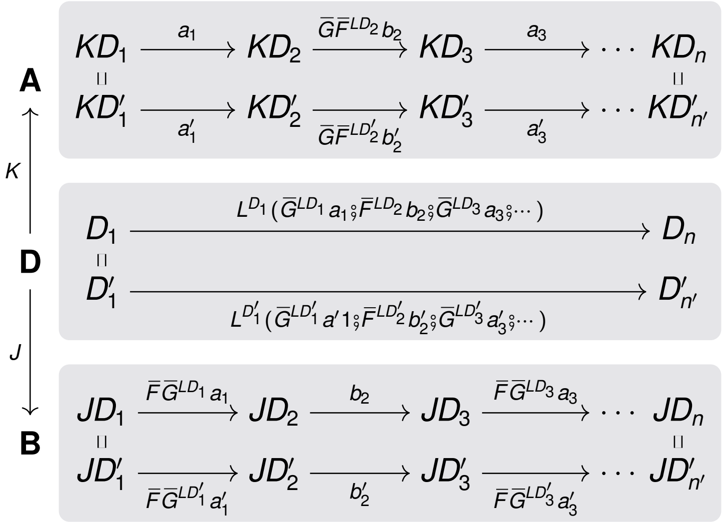
(K, J) *is independent*





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Sync minimality

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A lens span

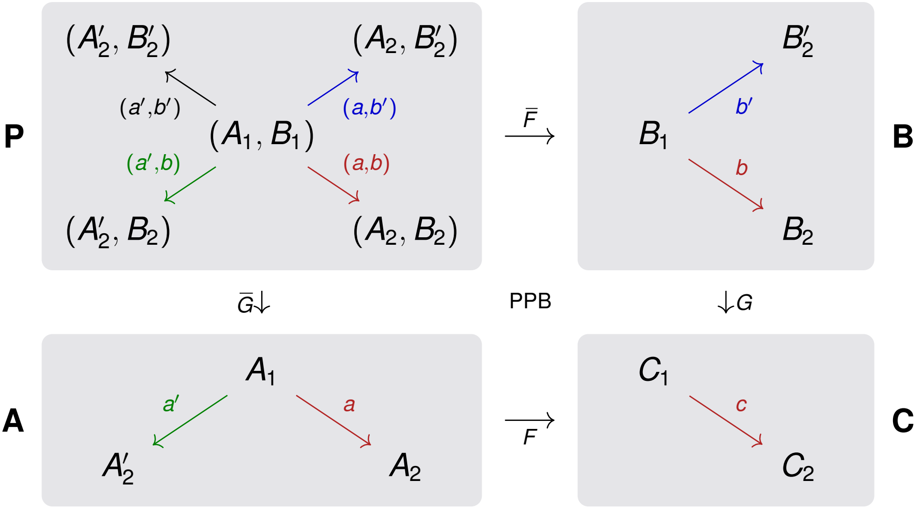
$$\mathbf{A} \xleftarrow{K} \mathbf{D} \xrightarrow{J} \mathbf{B}$$

is ***sync minimal*** if each morphism in **D** is a composite

$$D_1 \xrightarrow{d_1} D_2 \xrightarrow{d_2} D_3 \cdots D_{n-1} \xrightarrow{d_{n-1}} D_n$$

of lifts along K or J, that is, for each i, either $d_i = K^{D_i} K d_i$ or $d_i = J^{D_i} J d_i$.

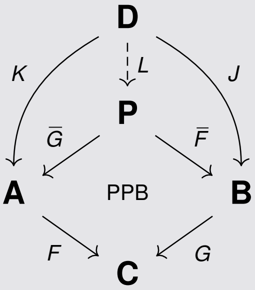
Non-example of sync minimality



Sufficient conditions

Proposition

If (K, J) is independent and is compatible with (F, G) and $(\overline{G}, \overline{F})$ is sync minimal, then a unique L exists.



Proof.

See my MRES thesis. □

Sync-minimal core

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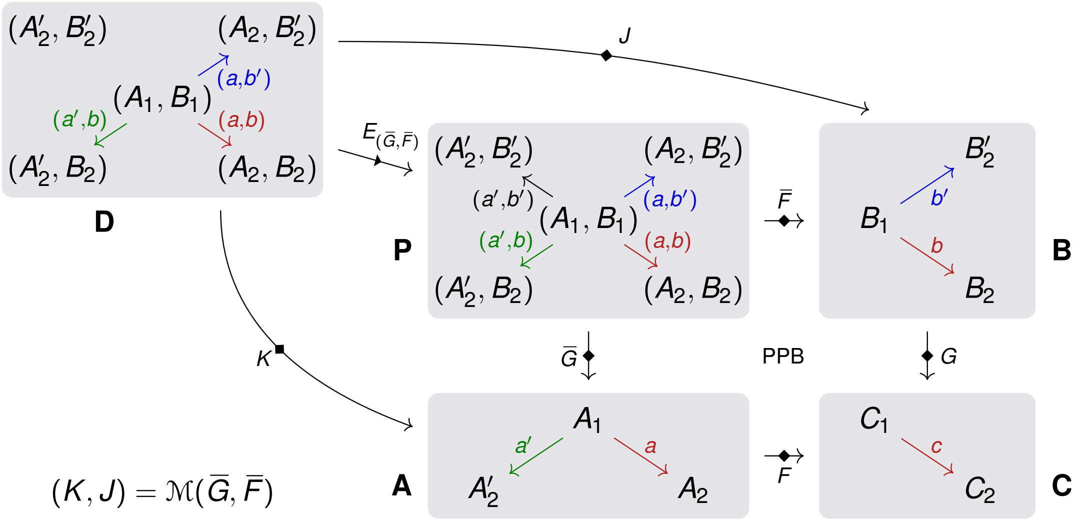
- The ***sync-minimal core*** of a lens span is obtained by removing all morphisms from its apex that are not composites of lifts along its legs.
- Let $\mathcal{M}(K, J)$ denote the sync-minimal core of a lens span (K, J)
- Let $E_{(K,J)}$ denote the inclusion of the apex of $\mathcal{M}(K, J)$ into that of (K, J)
- Independence of (K, J) is about morphisms in $\mathcal{M}(K, J)$

Example of sync-minimal core

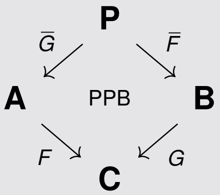


$$(K, J) = \mathcal{M}(\bar{G}, \bar{F})$$

Necessity of sync minimality

Proposition

If $(\overline{G}, \overline{F})$ is terminal amongst the independent spans that are compatible with (F, G), then $(\overline{G}, \overline{F})$ is sync minimal.



Proof sketch.

- $\mathcal{M}(\overline{G}, \overline{F})$ is independent and compatible with (F, G)
- There is a unique comparison lens H from $\mathcal{M}(\overline{G}, \overline{F})$ to $(\overline{G}, \overline{F})$
- As $(\mathcal{G}\overline{G}, \mathcal{G}\overline{F})$ is a pullback, $\mathcal{G}H = E_{(\overline{G}, \overline{F})}$
- H is surjective on morphisms as it is a surjective-on-objects lens
- H is actually the identity lens □

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When are proxy pullbacks real pullbacks?

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Proposition

A proxy pullback of a lens cospan is a real pullback if and only if it is sync minimal and all lens spans that form commuting squares with the cospan are independent and compatible with the cospan.

- Unsatisfactory as checking the independence and compatibility of ***all*** such lens spans is non-trivial
- Would be better if conditions were only in terms of the lens cospan

Discrete opfibrations and discrete categories

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Proposition

*A proxy pullback of a lens cospan is a real pullback **if** at least one leg of the cospan is a discrete opfibration.*

Proof.

See my MRES thesis. □

Proposition

*A proxy product of two categories is a real product **if and only if** at least one of them is a discrete category.*

Proof sketch.

For *only if* direction, the projection lenses of the funny tensor product of two non-discrete categories form a non-independent lens span. □

Conclusion

- Gave a new treatment of proxy pullbacks in terms of compatibility
- Characterised when a comparison lens to a proxy-pullback span exists
- Nicely characterised when proxy products are real products

Future work

- Nicely characterise when proxy pullbacks are real pullbacks
- Reformulate sync-minimality and independence at a higher level — e.g. the (cofaithful bijective-on-objects, cofull)-factorisation of the product pairing in $\mathcal{C}of$ of the put cofunctors of a lens span gives its sync-minimal core