RECALL:

Category $\mathbb{E}$ with pullbacks	Symmetric monoidal category $\underline{V}$ with equalisers preserved by functors of form $A\otimes(-)\otimes B$
$\mathbb{E}$	$\underline{\text{Comon}}_{\underline{V}}$
$\mathbb{E}/A$	$\underline{\text{Comod}}_V(A)$
$\Delta_f : \mathbb{E}/B \to \mathbb{E}/A$ pulls back along $f: A \to B$ in $\mathbb{E}$	$\Delta_f : \underline{\text{Comod}}_V(B) \to \underline{\text{Comod}}_V(A)$ is coinduction along $f: A \to B$ in $\underline{\text{Comon}}_V$
$\Sigma_f : \mathbb{E}/A \to \mathbb{E}/B$ composes with $f: A \to B$ in $\mathbb{E}$	$\Sigma_f : \underline{\text{Comod}}_V(A) \to \underline{\text{Comod}}_V(B)$ is corestriction along $f: A \to B$ in $\underline{\text{Comon}}_V$
$f: A \to B$ in $\mathbb{E}$ exponentiable if $\Delta_f \dashv \Pi_f$	$f: A \to B$ in $\underline{\text{Comon}}_V$ exponentiable if $\Delta_f \dashv \Pi_f$

A polynomial $p: I \to J$ is a diagram

$$I \xleftarrow{s} A \xrightarrow{f} B \xrightarrow{t} J$$

exponentiable

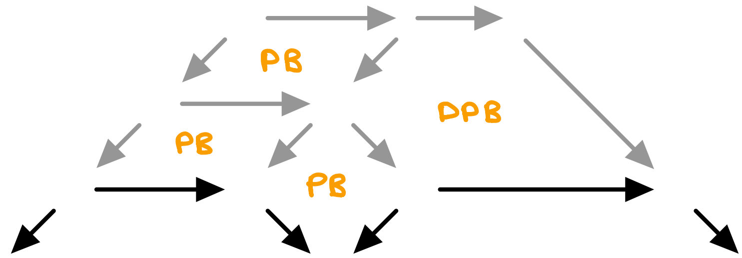
in $\mathbb{E}$ OR $\underline{\text{Comon}}_{\underline{V}}$.

The associated polynomial functor F_p is the composite:

$$\Sigma_t \Pi_f \Delta_s : \mathbb{E}/I \to \mathbb{E}/J \qquad \text{OR}$$

$$\Sigma_t \Pi_f \Delta_s : \underline{\text{Comod}}_V(I) \to \underline{\text{Comod}}_V(J)$$

Polynomials compose:



GOAL: Want mapping from polynomials to their associated functors to be functorial, i.e.

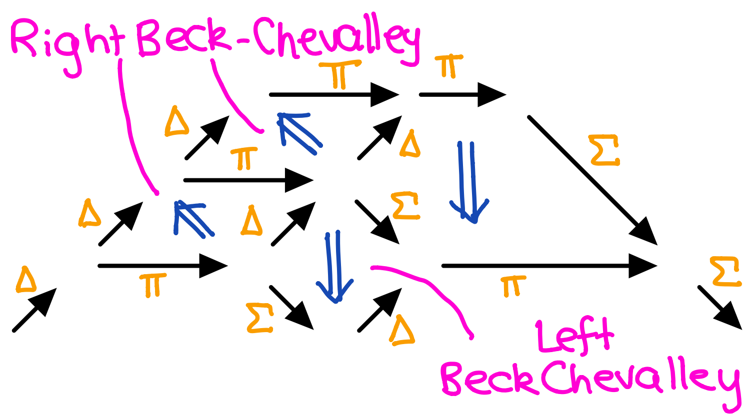
$$F_{pq} = F_p F_q$$

NEED:

(1) $\Sigma_{fg} \cong \Sigma_f \Sigma_g$, $\Delta_{fg} \cong \Delta_g \Delta_f$, $\Pi_{fg} \cong \Pi_f \Pi_g$

(2) exponentiable morphisms pullback stable and closed under composition

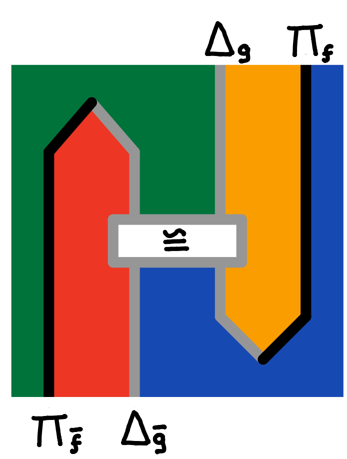
(3) canonical natural transformations below are isomorphisms



$$\begin{tikzcd} D \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Left Beck-Chevalley

Right Beck-Chevalley (f and $\bar{f}$ exponentiable)



PROP: Left Beck-Chevalley is iso $\iff$ Right Beck-Chevalley is iso

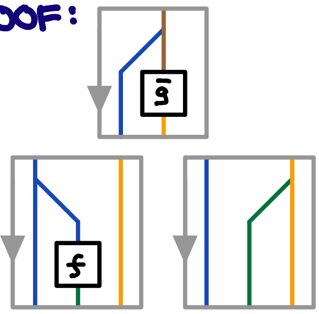
PROP:

MNAB is comodule pullback in $\underline{V}$ $\Longleftrightarrow$

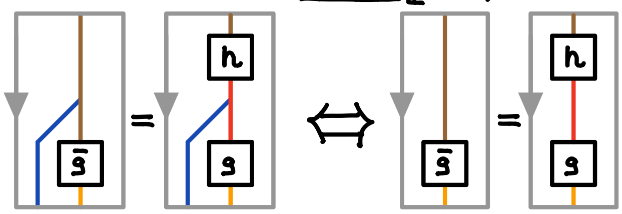
$$\begin{tikzcd} L \arrow[dr, "\exists ! h"] \arrow[drr, bend left, "\forall \bar{g}"] \arrow[ddr, bend right, dashed] & & \\ & m \arrow[r, "g"] \arrow[d, dashed] & N \arrow[d, dashed] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

(This is most of the proof that $\Sigma_f \dashv \Delta_f$)

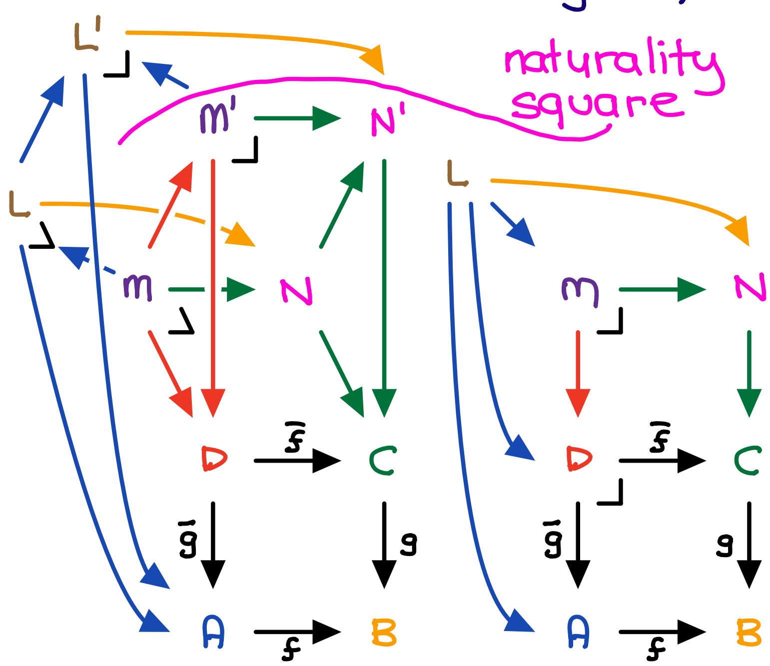
PROOF:

①  is fork in $\underline{\mathrm{Comod}}_V(A)$

② For all $h: L \to m$ in $\underline{\mathrm{Comod}}_V(A)$,



PROP (Left Beck–Chevalley in $\underline{E}$):



① Unique natural transformation

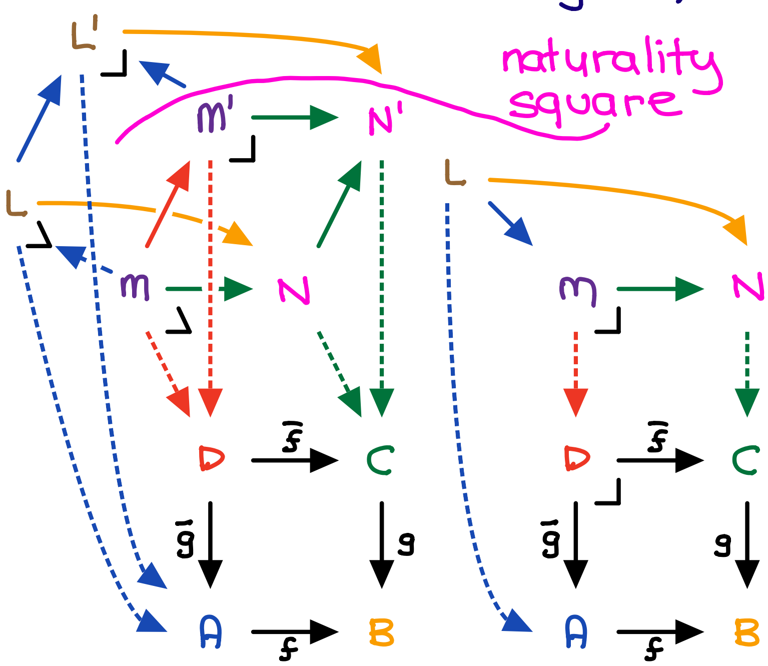
$\varphi : \Sigma_{\bar{g}} \Delta_{\bar{f}} \xrightarrow{\cdot} \Delta_f \Sigma_g$

② φ is iso $\Longleftrightarrow$ ABCD is pullback

($\Rightarrow$) take $N = (C, id_C)$

($\Leftarrow$) pullback pasting

PROP (Left Beck–Chevalley in $\underline{V}$):



① Unique natural transformation

$\varphi : \Sigma_{\bar{g}} \Delta_{\bar{f}} \xrightarrow{\cdot} \Delta_f \Sigma_g$

② φ is iso $\Longleftrightarrow$ ABCD is pullback

($\Rightarrow$) take $N = \langle C, \delta_C \rangle$

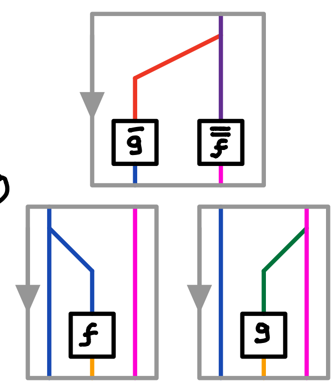
($\Leftarrow$) pullback pasting

Need fact that pullback in $\underline{\mathrm{Comon}}_V$ given by equaliser in $\underline{V}$

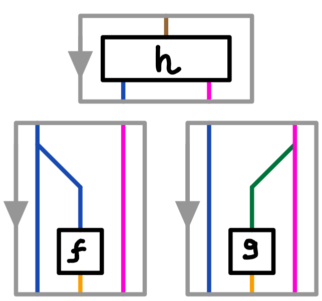
PROP (comodule-comonoid pullback pasting):

$$\begin{tikzcd} m \arrow[r, "\bar{\bar{f}}"] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, dashed] \\ D \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

PROOF:

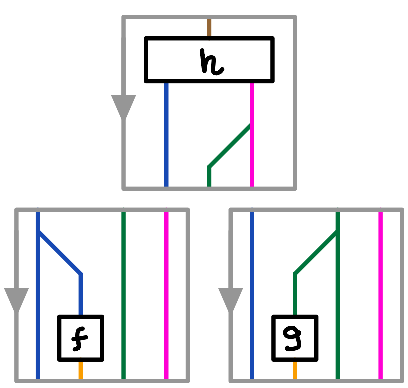
①  is a fork

Suppose that



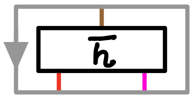
is a fork

②

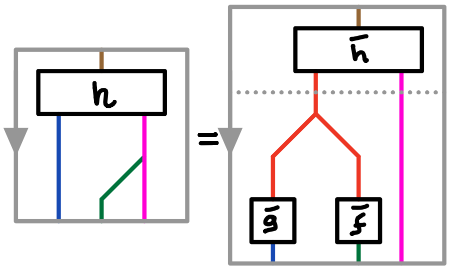


is a fork

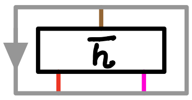
Exists unique

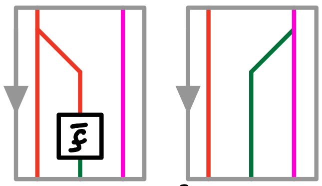


such that

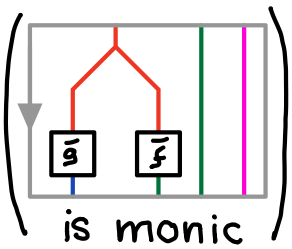


③





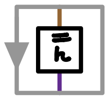
is a fork



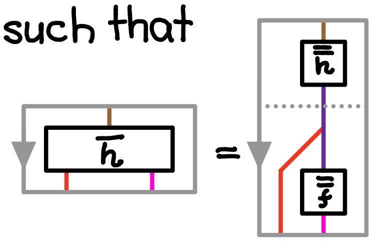
is monic

⑤ uniqueness

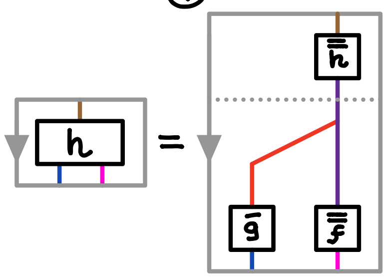
Exists unique



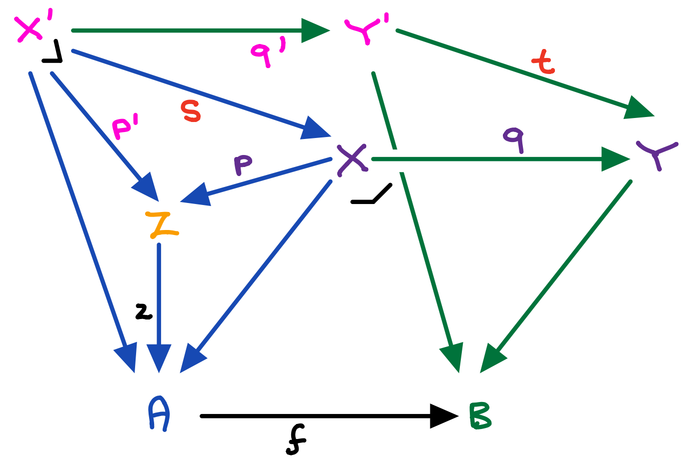
such that



④



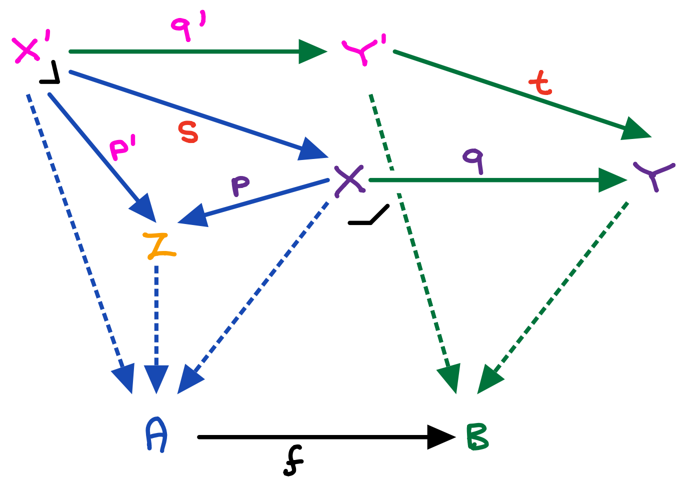
DEFN: distributivity pullback (or lax pullback complement) around $f: A \to B$ and $z: Z \to A$ is a terminal object in $\underline{PB}(f,z)$:



$(s,t): (X',Y',p',q') \longrightarrow (X,Y,p,q)$

REMARK: $\underline{PB}(f,z) \mathrel{\hat{=}} \Delta_f/(Z,z)$
A choice of distributivity pullback around f and z for each z above A gives a right adjoint to Δ_f and conversely.

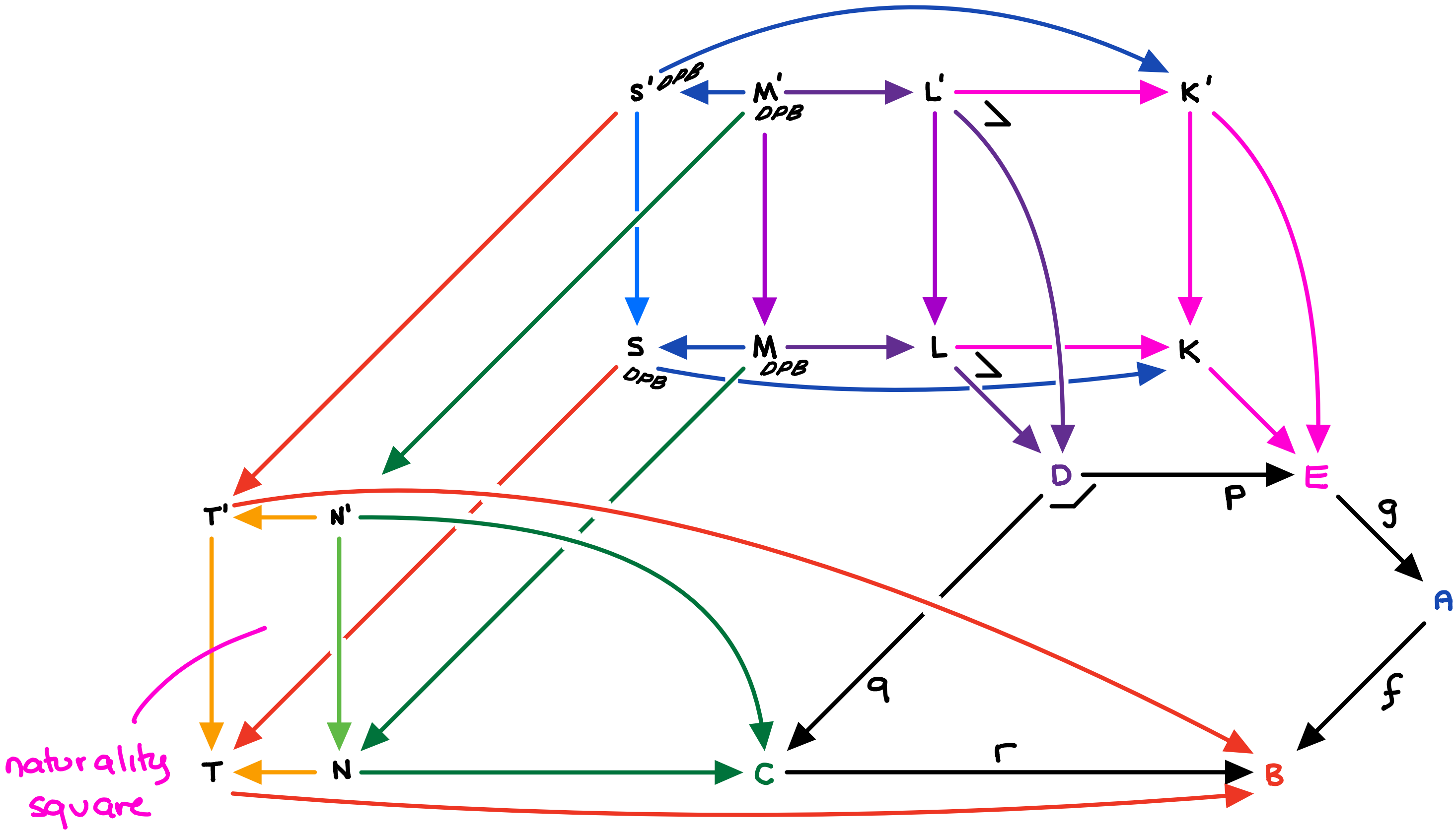
DEFN: generalised distributivity pullback around $f: A \to B$ (cocommutative comonoid morphism) and Z (A-comodule) is a terminal object in $\underline{CPB}(f,Z)$:



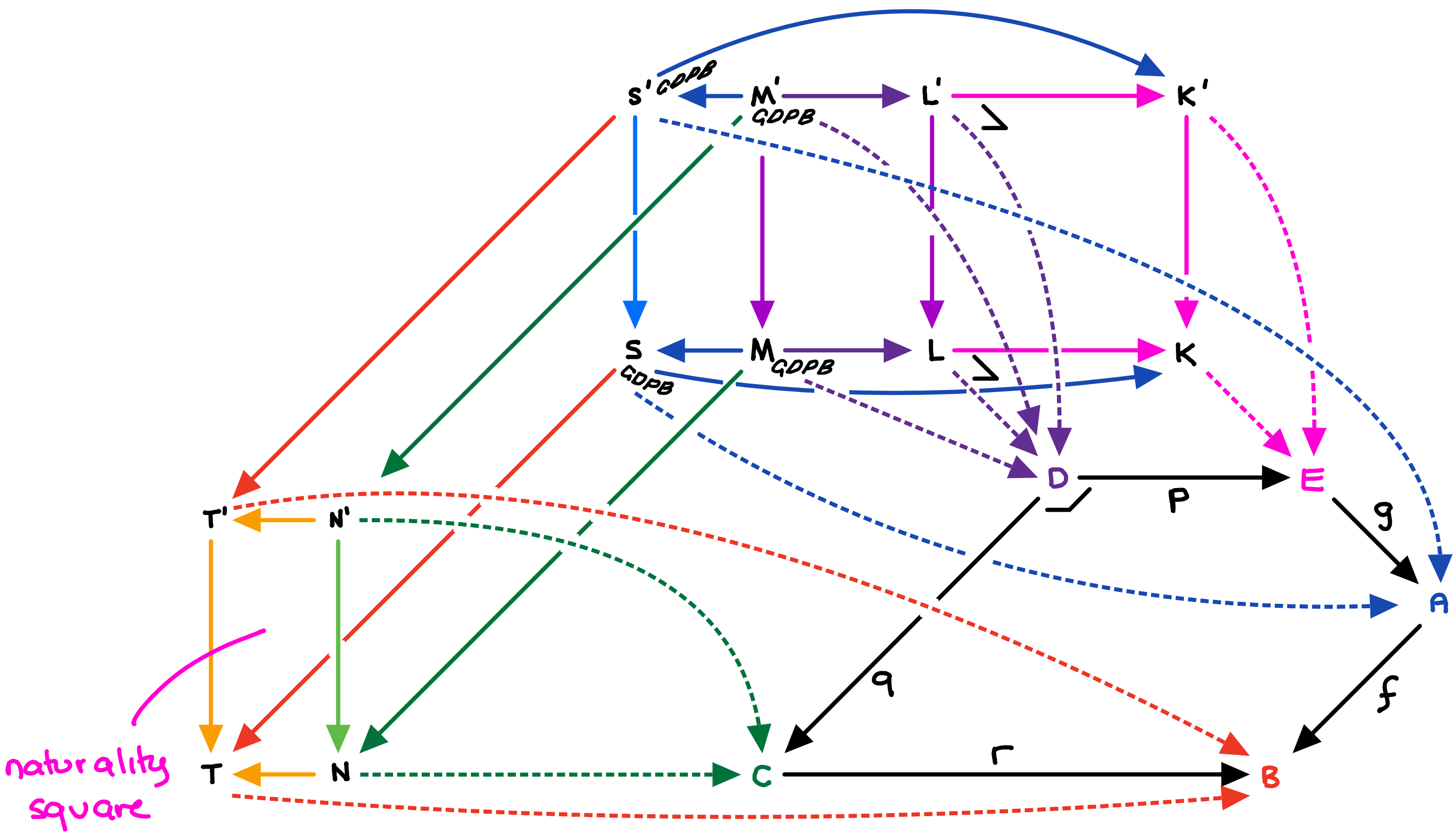
$(s,t): (X',Y',p',q') \longrightarrow (X,Y,p,q)$

REMARK: $\underline{CPB}(f,Z) \mathrel{\hat{=}} \Delta_f/Z$

PROP: in $\underline{E}$, unique natural transformation $\delta : \Sigma_r \Pi_q \Delta_p \stackrel{\cdot}{\longrightarrow} \Pi_f \Sigma_g$



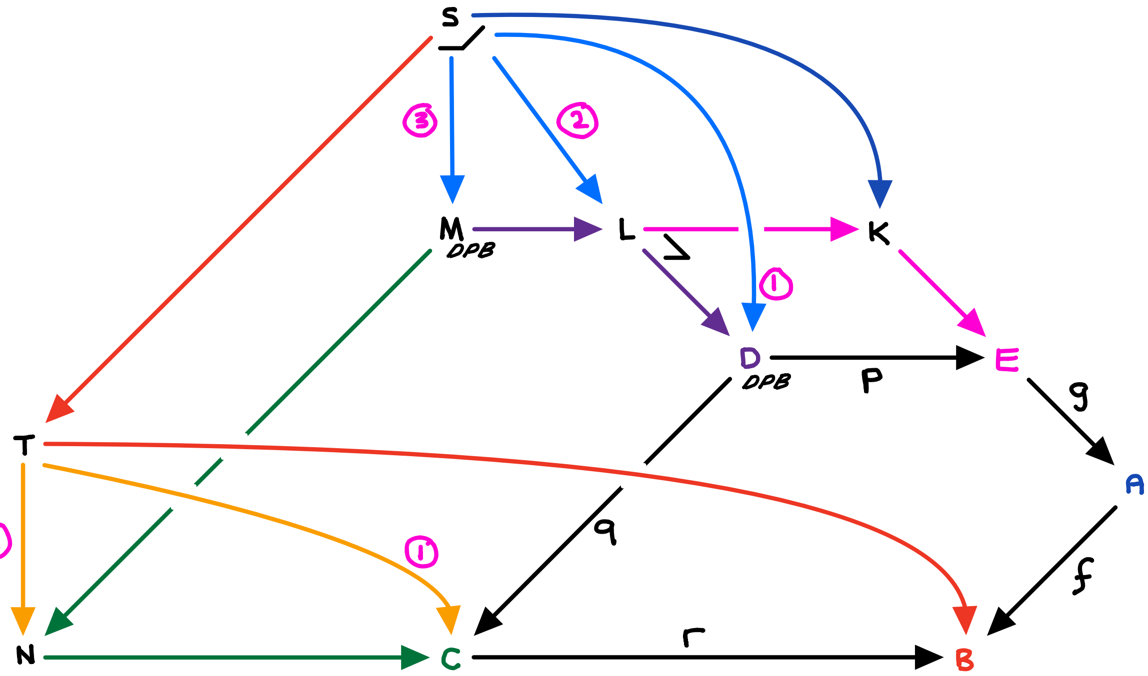
PROP: in $\underline{V}$, unique natural transformation $\delta : \Sigma_r \Pi_q \Delta_p \stackrel{\cdot}{\longrightarrow} \Pi_f \Sigma_g$



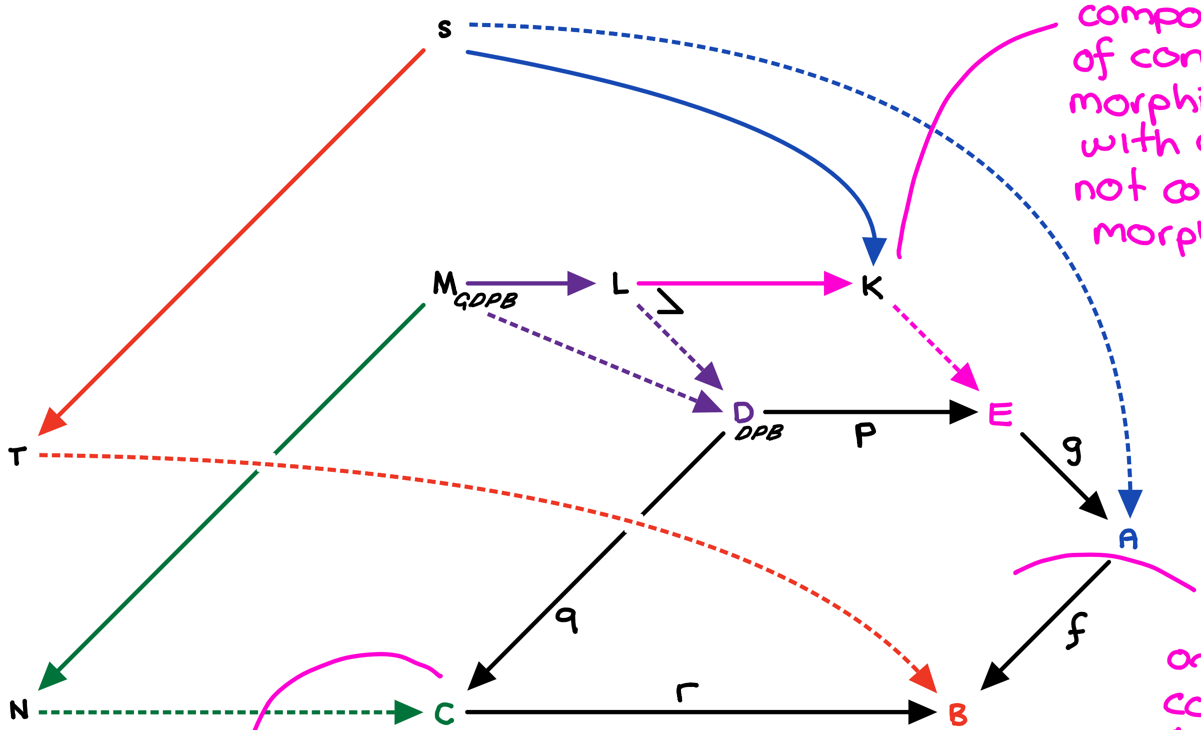
PROP: in $\underline{\mathbb{E}}$, $\delta : \Sigma_r \Pi_q \Delta_p \dashrightarrow \Pi_f \Sigma_g$ is iso iff ABCDE is a DPB

($\Rightarrow$) Take $K = \langle E, id_E \rangle$

($\Leftarrow$) Horizontal pasting of distributivity pullbacks



In $\underline{V}$, no obvious proof of horizontal pasting of GDPB with DPB



composite of comodule morphism with coaction not comodule morphism

DPB only about comonoids & comonoid morphisms

universal property gives comodule morphisms, not coactions