$\underline{E}$ = category with pullbacks

DEFN:
$$\begin{tikzcd} \underline{E}/A \arrow[r, shift left=2, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \underline{E}/B \arrow[l, shift left=2, "\Delta_f"] \end{tikzcd}$$

dependent sum

pullback

$$\begin{tikzcd} M \arrow[d, "m"'] \arrow[dr, "\Sigma_f(m)"] & \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} \bar{N} \arrow[r, "\varepsilon_n^{\Sigma f}"] \arrow[d, "\Delta_f(n)"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, "n"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

DEFN: f exponentiable if Δ_f has a right adjoint, denoted Π_f (dependent product)

DEFN: polynomial in $\underline{E}$

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[r, "f"] & B \arrow[dr, "t"] & \\ I & & & J \end{tikzcd}$$

exponentiable

DEFN: associated polynomial functor

$$\begin{tikzcd} & \underline{E}/A \arrow[r, "\Pi_f"] & \underline{E}/B \arrow[dr, "\Sigma_t"] & \\ \underline{E}/I \arrow[ur, "\Delta_s"] & & & \underline{E}/J \end{tikzcd}$$

EXAMPLE: $\underline{E} = \underline{Set}$, $I = J = 1$.

Given $b \in B$, $Y \in \underline{E}/A$, A_b-local sections of Y

$$\Pi_f(Y)_b = \left\{ h : \begin{tikzcd} A_b \arrow[rr, "h"] \arrow[dr] & & Y \arrow[dl] \\ & A & \end{tikzcd} \right\}$$

$$\underline{Set} \xrightarrow{\Delta_!} \underline{Set}/A \xrightarrow{\Pi_f} \underline{Set}/B \xrightarrow{\Sigma_!} \underline{Set}$$

$$S \longmapsto \begin{matrix} S\times A \\ \pi_2 \downarrow \\ A \end{matrix} \longmapsto \begin{matrix} \sum_{b\in B} S^{A_b} \\ \downarrow \\ B \end{matrix} \longmapsto \sum_{b\in B} S^{A_b}$$

Weber (2015) "Polynomials in Categories with Pullbacks"

$\underline{V}$ = symmetric monoidal category with equalisers, such that all functors $A \otimes (-) \otimes B$ preserve them

EXAMPLE: $\underline{V}$ is cartesian monoidal with equalisers (i.e. finitely complete)

EXAMPLE: $\underline{V} = \underline{Vect}_{\mathbb{K}}$. From

(1) $Eq(T_1, T_2) = Ker(T_1 - T_2)$ for $T_1, T_2 : V \to W$

(2) $Ker(T_1 \otimes T_2) = Ker(T_1) \otimes W + V \otimes Ker(T_2)$, for $T_1 : V \to V'$, $T_2 : W \to W'$

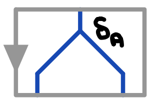
we have

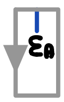
$$ker(X \otimes T_1 \otimes Y - X \otimes T_2 \otimes Y) = X \otimes ker(T_1 - T_2) \otimes Y,$$

so $X \otimes (-) \otimes Y$ preserves equalisers.

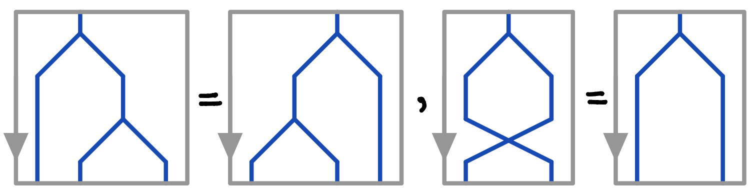
EXAMPLE: $\underline{V} = \underline{Mon}_{\underline{U}}$ for any $\underline{U}$ satisfying the above conditions.

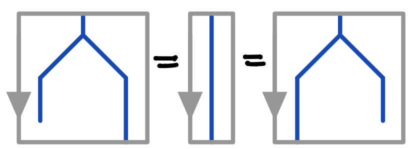
DEFN: $\underline{Comon}_{\underline{V}}$ = category of cocommutative comonoids

OBJECTS: (A , 

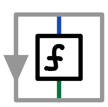
, 

) such that

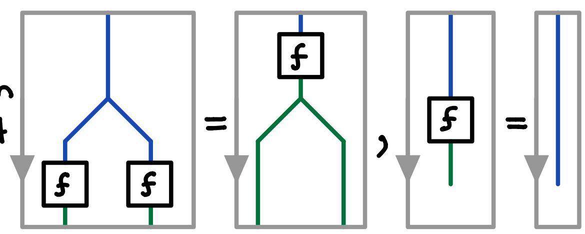


and 

MORPHISMS $f : (A, \delta_A, \varepsilon_A) \to (B, \delta_B, \varepsilon_B)$:

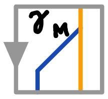


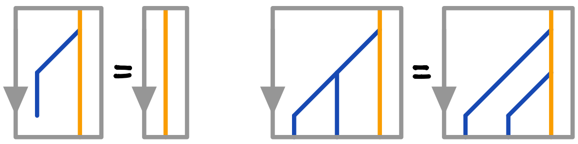
such that



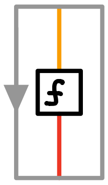
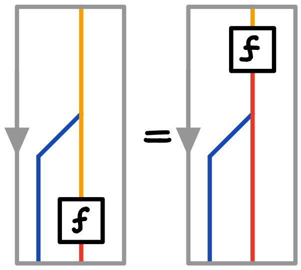
Aguiar (1997) "Internal Categories and Quantum Groups"

DEFN: $\underline{\mathrm{Comod}}_{\underline{\mathcal V}}(A)$ = category of A-comodules

OBJECTS: $\Big(m,$  $\Big)$ such that



MORPHISMS $f:(m,\gamma_m)\to(N,\gamma_N)$:

 such that 

EXAMPLE: if $\underline{\mathcal V}$ is cartesian monoidal, $\underline{\mathrm{Comon}}_{\underline{\mathcal V}}\cong\underline{\mathcal V}$ and $\underline{\mathrm{Comod}}_{\underline{\mathcal V}}(A)\cong\underline{\mathcal V}/A$

EXAMPLES: $\underline{\mathcal V}=\mathrm{Vect}_{\mathbb K}$

(1) Group-like coalgebra on a set S:

$$\big(\mathbb K S,\ \delta(s)=s\otimes s,\ \varepsilon(s)=1\big)$$

$\mathbb K S$-comodules $M \leftrightsquigarrow$ Families $\{M_s\}_{s\in S}$ of $\mathbb K$-spaces

$$M_s=\{m\in M:\ \gamma_M(m)=s\otimes m\}.$$

$$M=\bigoplus_{s\in S}M_s \qquad \gamma_M(m)=\{\,s\otimes m \ \text{ if } m\in M_s$$

(2) Divided power coalgebra C:

$$\Big(\mathbb K\mathbb N,\ \delta(n)=\sum_{j=0}^{n} j\otimes(n-j),\ \varepsilon(n)=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n>0\end{cases}\Big)$$

Dual algebra is $\mathbb K[[X]]$: $a\in C^*\rightsquigarrow\sum_{j=0}^{\infty}a(j)X^j$

$\underline{\mathrm{Comod}}(C)$ is the full subcategory of $\underline{\mathrm{Mod}}(\mathbb K[[X]])$ of torsion $\mathbb K[[X]]$-modules.

(3) Trigonometric coalgebra $T_{\mathbb K}$:

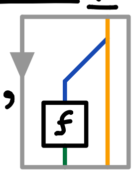
$$\Big(\mathbb K\{s,c\},\ \overset{\delta}{\begin{matrix}c\mapsto c\otimes c-s\otimes s\\ s\mapsto s\otimes c+c\otimes s\end{matrix}},\ \overset{\varepsilon}{\begin{matrix}c\mapsto 1\\ s\mapsto 0\end{matrix}}\Big)$$

Dual algebras: $T^*_{\mathbb R}\cong\mathbb C$, $T^*_{\mathbb C}\cong\mathbb H$

$\underline{\mathrm{Comod}}(T_{\mathbb K})\cong\underline{\mathrm{Mod}}(T^*_{\mathbb K})$

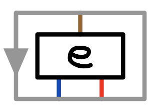
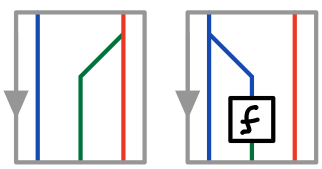
Lin (1975) "Semiperfect Coalgebras" p363

DEFN: given $f:A\to B$ in $\underline{\mathrm{Comon}}_{\underline{\mathcal V}}$

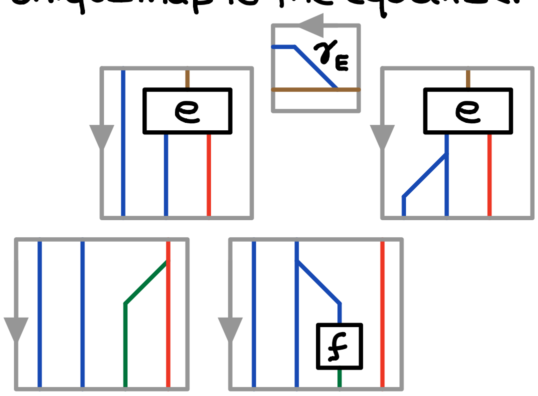
$m \longmapsto \Big(m,$  $\Big)$

$$\begin{tikzcd}
\underline{\mathrm{Comod}}_{\underline{\mathcal V}}(A) \arrow[r, shift left=2, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Comod}}_{\underline{\mathcal V}}(B) \arrow[l, shift left=2, "\Delta_f"]
\end{tikzcd}$$

$(E,\gamma_E) \longleftarrow\!\!| \ N$

 equalises  in $\underline{\mathcal V}$,

γ_E is unique map to the equaliser:



By the diagram

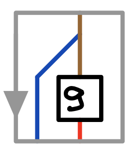
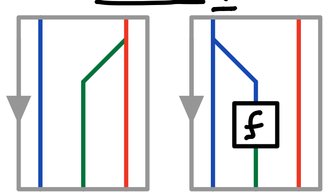
$$\begin{tikzcd}
E \arrow[r, "g"] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, dashed] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

we mean that:

(1) $f:A\to B$ is in $\underline{\mathrm{Comon}}_{\underline{\mathcal V}}$

(2) $E\in\underline{\mathrm{Comod}}_{\underline{\mathcal V}}(A)$

(3) $g:\Sigma_f(E)\to N$ is in $\underline{\mathrm{Comod}}_{\underline{\mathcal V}}(B)$

(4)  equalises 

DEFN: f is exponentiable if Δ_f has a right adjoint, denoted Π_f.

DEFN: polynomial in $\underline{V}$

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[r, "f"] & B \arrow[dr, "t"] & \\ I & & & J \end{tikzcd}$$

exponentiable

in $\underline{\mathrm{Comon}}_{\underline{V}}$

DEFN: associated polynomial functor

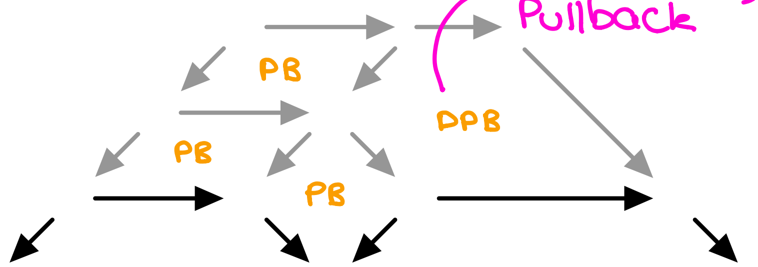
$$\begin{tikzcd} & \underline{\mathrm{Comod}}_{\underline{V}}(A) \arrow[r, "\Pi_f"] & \underline{\mathrm{Comod}}_{\underline{V}}(B) \arrow[dr, "\Sigma_t"] & \\ \underline{\mathrm{Comod}}_{\underline{V}}(I) \arrow[ur, "\Delta_s"] & & & \underline{\mathrm{Comod}}_{\underline{V}}(J) \end{tikzcd}$$

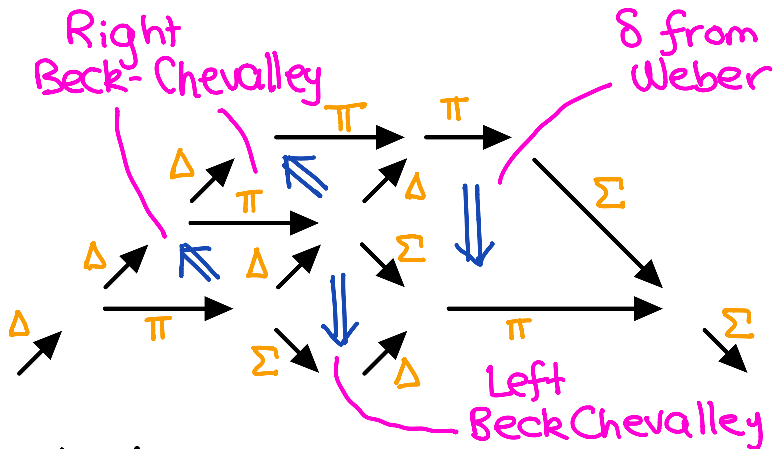
REMARK: reduces to earlier case if $\underline{V}$ cartesian monoidal

REMARK: $\underline{V}$ is $\underline{\mathrm{Comon}}_{\underline{V}}$-indexed. A-comodule = A-indexed family in $\underline{V}$

Grunenfelder and Paré (1987) "Families Parametrized by Coalgebras"

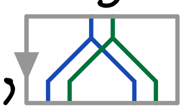
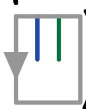
Composition of polynomials in $\underline{E}$



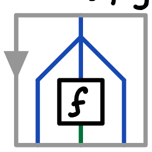
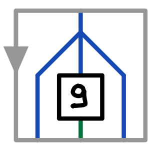


Need
(1) $\Sigma_{(-)}$, $\Delta_{(-)}$, $\Pi_{(-)}$ functorial
(2) exponentiable pullback stable and closed under composition
(3) canonical cells to be isos

PROP: $\underline{\mathrm{Comon}}_{\underline{V}}$ is finitely complete

PRODUCT OF A, B: $\big(A \otimes B,$  ,  $\big)$

EQUALISER OF $f, g : A \to B$:

equaliser of  and  in $\underline{V}$.

TERMINAL: I with structure morphisms

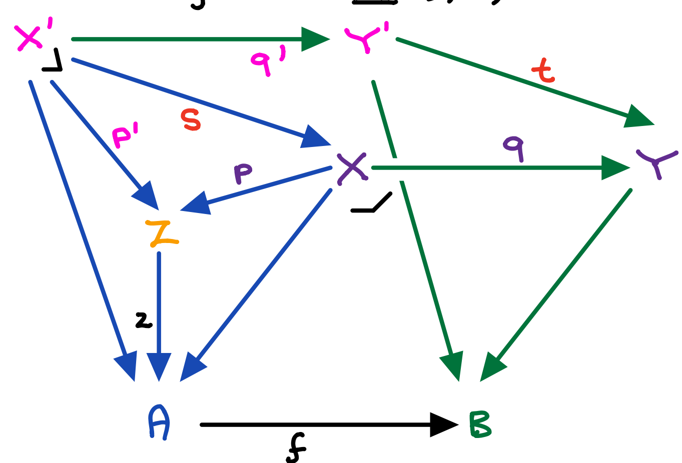
PULLBACK OF $A \xrightarrow{f} B \xleftarrow{g} C$: $\Delta_f \Sigma_g(C)$

REMARK: for (not-necessarily cocommutative) comonoids, $\otimes$ is a symmetric monoidal product with projections; also equalisers and terminal given as above. The equaliser of $\pi_A ; f$ and $\pi_C ; g$ is the relative pullback of f and g, and only sometimes coincides with $\Delta_f \Sigma_g(C)$

DEFN: polynomials in $\underline{V}$ compose as in $\underline{E}$ *

Bohm (2018) "Crossed Modules of Monoids, I"

DEFN: distributivity pullback (or lax pullback complement) around $f : A \to B$ and $z : Z \to A$ is a terminal object in $\underline{PB}(f, z)$:



$(s, t) : (X', Y', p', q') \longrightarrow (X, Y, p, q)$

REMARK: $\underline{PB}(f, z) \cong \Delta_f / (Z, z)$
A choice of distributivity pullback around f and z for each z above A gives a right adjoint to Δ_f and conversely.

Hoesseini, Tholen and Yeganeh (2018) "Lax Pullback Complements and Pullbacks of Spans"

DEFN: *generalised distributivity pullback* around $f : A \to B$ (cocommutative comonoid morphism) and Z (A-comodule) is a terminal object in $\underline{\mathrm{GPB}}(f, Z)$:

$$\begin{tikzcd}
X' \arrow[rrr, "q'"] \arrow[drrr, "s"] \arrow[ddr, "p'"'] \arrow[dddr, dashed] & & & Y' \arrow[drr, "t"] \arrow[dddr, dashed] & & \\
& & & X \arrow[rr, "q"] \arrow[dll, "p"'] \arrow[ddll, dashed] & & Y \arrow[ddl, dashed] \\
& Z \arrow[d, dashed] & & & & \\
& A \arrow[rrr, "f"'] & & & B &
\end{tikzcd}$$

$$(s, t) : (X', Y', p', q') \longrightarrow (X, Y, p, q)$$

REMARK: $\underline{\mathrm{GPB}}(f, Z) \cong \Delta_f / Z$