M*-CATEGORIES

MATTHEW DI MEGLIO
(Joint work with Chris Heunen)

AUSTRALIAN CATEGORY THEORY SEMINAR
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Starting point

AXIOMS FOR THE CATEGORY OF HILBERT SPACES

CHRIS HEUNEN AND ANDRE KORNELL

ABSTRACT. We provide axioms that guarantee a category is equivalent to that of continuous linear functions between Hilbert spaces. The axioms are purely categorical and do not presuppose any analytical structure. This addresses a question about the mathematical foundations of quantum theory raised in reconstruction programmes such as those of von Neumann, Mackey, Jauch, Piron, Abramsky, and Coecke.

Quantum mechanics has mathematically been firmly founded on Hilbert spaces and operators between them for nearly a century [32]. There has been continuous inquiry into the special status of this foundation since [26, 8, 16]. How are the mathematical axioms to be interpreted physically? Can the theory be reconstructed from a different framework whose axioms can be interpreted physically? Such reconstruction programmes involve a mathematical reformulation of (a generalisation of) the theory of Hilbert spaces and their operators, such as operator algebras [23], orthomodular lattices [13, 25], and, most recently, categorical quantum mechanics [1, 5]. The latter uses the framework of category theory [19], and emphasises operators more than their underlying Hilbert spaces. It postulates a category with structure that models physical features of quantum theory [12]. The question of how "to justify the use of Hilbert space" [25] then becomes: which axioms guarantee that a category is equivalent to that of continuous linear functions between Hilbert spaces? This article answers that mathematical question. The axioms are purely categorical in nature, and do not presuppose any analytical structure such as continuity, complex numbers, or probabilities. The approach is similar to Lawvere's categorical characterisation of the theory of sets [17].

A characterisation for the category of Hilbert spaces

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Abstract

The categories of real and of complex Hilbert spaces with bounded linear maps have received purely categorical characterisations by Chris Heunen and Andre Kornell. These characterisations are achieved through Solèr's theorem, a result which shows that certain orthomodularity conditions on a Hermitian space over an involutive division ring result in a Hilbert space with the division ring being either the reals, complexes or quaternions. The characterisation by Heunen and Kornell makes use of a monoidal structure, which in turn excludes the category of quaternionic Hilbert spaces. We provide an alternative characterisation without the assumption of monoidal structure on the category. This new approach not only gives a new characterisation of the categories of real and of complex Hilbert spaces, but also the category of quaternionic Hilbert spaces.

$$\boxed{\begin{gathered} 1^* = 1 \\ (fg)^* = g^* f^* \\ f^{**} = f \end{gathered}}$$

Let $\mathbf{C}$ be a $*$-category with

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$(fg)^* = g^* f^*$
$f^{**} = f$

Let $\mathbf{C}$ be a $*$-category with

(1) a zero object

(2) binary orthonormal biproducts, and

$i_1 = p_1^*$
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(3) isometric equalisers, such that

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If $\mathbf{C}$ has a simple separator A then $\mathbf{C}(A,A)$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and

$$\begin{tikzcd} & \mathbf{Hilb}_{\mathbf{C}(A,A)} \arrow[d] \\ \mathbf{C} \arrow[ur, dashed, "\simeq"] \arrow[r, "\mathbf{C}(A,-)"'] & \mathbf{Set} \end{tikzcd}$$

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Let $\mathbf{C}$ be a $*$-category with

addition

orthogonal complements

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adjoints multiplication

Let $\mathbf{C}$ be a $*$-category with

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(2) binary orthonormal biproducts, and

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negation

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Want to understand the directed colimit axiom

Will adapt ideas from

DAGGER CATEGORIES AND THE COMPLEX NUMBERS: AXIOMS FOR THE CATEGORY OF FINITE-DIMENSIONAL HILBERT SPACES AND LINEAR CONTRACTIONS

MATTHEW DI MEGLIO AND CHRIS HEUNEN

ABSTRACT. We characterise the category of finite-dimensional Hilbert spaces and linear contractions using simple category-theoretic axioms that do not refer to norms, continuity, dimension, or real numbers. Our proof directly relates limits in category theory to limits in analysis, using a new variant of the classical characterisation of the real numbers instead of Solèr's theorem.

1. INTRODUCTION

The category **Hilb** of Hilbert spaces and bounded linear maps and the category **Con** of Hilbert spaces and linear contractions were both recently characterised in terms of simple category-theoretic structures and properties [6, 7]. For example, the structure of a *dagger* encodes adjoints of linear maps. Remarkably, none of

Codirected limits in $\mathbf{Hilb}_{\leqslant 1}$

complex
Hilbert spaces
and contractions

$\|fx\| \leqslant \|x\|$

Codirected limits in $\mathbf{Hilb}_{\leqslant 1}$

complex Hilbert spaces and contractions

$\|fx\| \leqslant \|x\|$

$\alpha, \beta \in J$

$\alpha \leqslant \beta$

$$X_\alpha \xleftarrow{f_{\alpha \leqslant \beta}} X_\beta$$

Codirected limits in $\mathbf{Hilb}_{\leqslant 1}$

complex Hilbert spaces and contractions

$\|fx\| \leqslant \|x\|$

$\alpha, \beta \in J$

$\alpha \leqslant \beta$

$$\begin{tikzcd} X_\alpha & X_\beta \arrow[l, "f_{\alpha\leqslant\beta}"] & X \arrow[l, bend right, "f_\beta"'] \arrow[ll, bend right=40, "f_\alpha"'] \end{tikzcd}$$

$$X = \left\{ x \in \prod_{\alpha \in J} X_\alpha \;\middle|\; f_{\alpha\leqslant\beta}\, x_\beta = x_\alpha \text{ and } \sup_{\alpha\in J} \|x_\alpha\|^2 < \infty \right\}$$

$$f_\alpha x = x_\alpha \qquad \langle x | y \rangle = \lim_{\alpha \in J} \langle x_\alpha | y_\alpha \rangle$$

Codirected limits in $\mathbf{Hilb}_{\leqslant 1}$

complex Hilbert spaces and contractions

$\|fx\| \leqslant \|x\|$

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$$\begin{tikzcd}
& & A \arrow[d, dashed, "g"] \arrow[dll, bend right, "g_\alpha"'] \arrow[dl, bend right, "g_\beta"'] \\
& & X \arrow[dll, "f_\alpha"'] \arrow[dl, "f_\beta"] \\
X_\alpha & X_\beta \arrow[l, "f_{\alpha\leqslant\beta}"] &
\end{tikzcd}$$

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Bounded codirected limits of contractions in **Hilb**

$$g_\alpha^* g_\alpha \leqslant g_\beta^* g_\beta \leqslant b$$

$\alpha, \beta \in J$

$\alpha \leqslant \beta$

$$\begin{tikzcd}
 & & A \arrow[dll, bend right, "g_\alpha"'] \arrow[dl, bend right, "g_\beta"'] \arrow[d, dashed, "g"] \\
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$\mathbf{C}(A,A)$ is monotone complete

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Need directed colimits in $\mathbf{C}_{\leqslant 1}$ rather than $\mathbf{C}_1$

$$\begin{tikzcd}
& & A \arrow[d, dashed, "g"] \arrow[ddll, bend right, "g_\alpha"'] \arrow[ddl, bend right, "g_\beta"] \\
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X_\alpha & X_\beta \arrow[l, "f_{\alpha\leqslant\beta}"] &
\end{tikzcd}$$

$$g^*g = \sup_{\alpha \in J} g_\alpha^* g_\alpha$$

What about completeness of arbitrary homsets?

DEFINITION:

An order sum of an orthogonal family $(x_\alpha)_{\alpha \in J}$ of elements of an inner product module is an element x such that

(i) $\langle x | x \rangle = \sup\limits_{F \subseteq_{\text{fin}} J} \sum\limits_{\alpha \in F} \langle x_\alpha | x_\alpha \rangle$ (ii) $\langle x | x_\alpha \rangle = \langle x_\alpha | x_\alpha \rangle$

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PROPOSITION:

Order sums are unique

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DEFINITION:

An inner product module is orthogonally complete if $(x_\alpha)_{\alpha \in J}$ is order summable whenever exists $b \geq 0$ such that

$$\sum_{\alpha \in F} \langle x_\alpha | x_\alpha \rangle \leq b \qquad \text{for all finite } F \subseteq J.$$

PROPOSITION: $\mathbf{C}(A, X)$ is orthogonally complete

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$$A \xrightarrow{\quad f_\alpha \quad} X$$

$$f_\alpha^* f_\beta = 0$$

$$b \geqslant \sum_{\alpha \in F} f_\alpha^* f_\alpha$$

PROPOSITION: $\mathbf{C}(A, X)$ is orthogonally complete

$$\begin{tikzcd} A \arrow[rr, "f_\alpha"] \arrow[dr, two heads, "g_\alpha"'] & & X \\ & Y_\alpha \arrow[ur, tail, "m_\alpha"'] & \end{tikzcd}$$

$f_\alpha^* f_\beta = 0$

g_α^* monic

g_β epic $\qquad m_\alpha^* m_\alpha = 1$

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```tikzcd
A \arrow[rr, "f_\alpha"] \arrow[dr, two heads, "g_\alpha"'] & & X \\
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```

$$\left.\begin{array}{r} f_\alpha^* f_\beta = 0 \\ g_\alpha^* \text{ monic} \\ g_\beta \text{ epic} \end{array}\right\} \Rightarrow \begin{array}{l} m_\alpha^* m_\beta = 0 \\ m_\alpha^* m_\alpha = 1 \end{array}$$

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A \arrow[dr, "g_F"'] & & X \\
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b \geq \sum_{\alpha\in F} f_\alpha^* f_\alpha = \sum_{\alpha\in F} g_\alpha^* g_\alpha = g_F^* g_F
$$

Proposition: $\mathbf{C}(A,X)$ is orthogonally complete

$$\begin{tikzcd}
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$$\begin{tikzcd}
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We assumed

(5) $\mathbf{C}_1$ has directed colimits.

but used

(5') $\mathbf{C}_{\leqslant 1}$ has directed colimits.

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PROPOSITION: (5) and (5') are equivalent.

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A codilator of a morphism $f : X \to Y$ is an initial codilation of f

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$$\begin{tikzcd}[column sep=small] & X \boxplus_f Y & \\ X \arrow[ur, tail, "i_1"'] & & Y \arrow[ul, tail, "i_2"] \end{tikzcd}$$

$$i_1^* i_1 = 1 \qquad i_2^* i_1 = f \qquad i_2^* i_2 = 1$$

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$$
\begin{tikzcd}
 & & S & & \\
 & & X \boxplus_f Y & & \\
X \arrow[uurr, bend left, "s_1"] \arrow[urr, "i_1"'] & & & & Y \arrow[uull, bend right, "s_2"'] \arrow[ull, "i_2"]
\end{tikzcd}
$$

$$s_1^* s_1 = 1 \qquad s_2^* s_1 = f \qquad s_2^* s_2 = 1$$

$$i_1^* i_1 = 1 \qquad i_2^* i_1 = f \qquad i_2^* i_2 = 1$$

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$$\begin{tikzcd}
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& X \boxplus_f Y \arrow[u, dashed, "{[s_1\ s_2]}"] & \\
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(like a cotabulation in an allegory)

LEMMA: If $\mathbf{C}$ is an M*-category, then $\mathbf{C}_1 \to \mathbf{C}_{\leq 1}$ preserves directed colimits

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$$
\begin{tikzcd}
 & & & A \\
X_\alpha \arrow[urrr, bend left] \arrow[r, tail, "f_{\alpha\leqslant\beta}"'] & X_\beta \arrow[urr, bend left] & \cdots \arrow[r, tail] & X
\end{tikzcd}
$$

LEMMA: If $\mathbf{C}$ is an M^*-category, then $\mathbf{C}_1 \to \mathbf{C}_{\leq 1}$ preserves directed colimits

$$
\begin{tikzcd}
 & & & & A \arrow[dllll, tail, bend right, "t_\alpha"'] \arrow[dll, tail, bend right, "t_\beta"] \\
Y_\alpha & & Y_\beta & & \\
X_\alpha \arrow[u, tail, "s_\alpha"] \arrow[rr, tail, "f_{\alpha \leq \beta}"'] & & X_\beta \arrow[u, tail, "s_\beta"] & \cdots \arrow[r, tail] & X
\end{tikzcd}
$$

LEMMA: If $\mathbf{C}$ is an M^*-category, then $\mathbf{C}_1 \to \mathbf{C}_{\leqslant 1}$ preserves directed colimits

$$
\begin{tikzcd}
 & & & A \arrow[lll, bend right, tail, "t_\alpha"'] \arrow[ll, bend right, tail, "t_\beta"] \\
Y_\alpha \arrow[r, tail, "g_{\alpha\leq\beta}"'] & Y_\beta & & \\
X_\alpha \arrow[u, tail, "s_\alpha"] \arrow[r, tail, "f_{\alpha\leq\beta}"'] & X_\beta \arrow[u, tail, "s_\beta"] & \cdots \arrow[r, tail] & X
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$$
\begin{tikzcd}
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X_\alpha \arrow[u, tail, "s_\alpha"] \arrow[r, tail, "f_{\alpha\leq\beta}"'] & X_\beta \arrow[u, tail, "s_\beta"] & \cdots \arrow[r, tail] & X
\end{tikzcd}
$$

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$$
\begin{tikzcd}
 & & & & A \arrow[dllll, bend right, tail, "|" marking, "t_\alpha"'] \arrow[dlll, bend right, tail, "|" marking, "t_\beta"] \\
Y_\alpha \arrow[r, tail, "|" marking, "g_{\alpha\leq\beta}"'] & Y_\beta & \cdots \arrow[r, tail, "|" marking] & Y & \\
X_\alpha \arrow[u, tail, "|" marking, "s_\alpha"] \arrow[r, tail, "|" marking, "f_{\alpha\leq\beta}"'] & X_\beta \arrow[u, tail, "|" marking, "s_\beta"] & \cdots \arrow[r, tail, "|" marking] & X \arrow[u, tail, "|" marking, "s"'] &
\end{tikzcd}
$$

LEMMA: If $\mathbf{C}$ is an M^*-category, then $\mathbf{C}_1 \to \mathbf{C}_{\leq 1}$ preserves directed colimits

$$
\begin{tikzcd}
 & & & & A \arrow[llll, tail, bend right, "t_\alpha"'] \arrow[lll, tail, bend right, "t_\beta"] \arrow[d, tail, "t"] \\
Y_\alpha \arrow[r, tail, "g_{\alpha\leq\beta}"'] & Y_\beta & \cdots \arrow[rr, tail] & & Y \\
X_\alpha \arrow[r, tail, "f_{\alpha\leq\beta}"'] \arrow[u, tail, "s_\alpha"] & X_\beta \arrow[u, tail, "s_\beta"] & \cdots \arrow[rr, tail] & & X \arrow[u, tail, "s"']
\end{tikzcd}
$$

PROPOSITION:

Every strict contraction has a codilator

$\boxed{\begin{array}{c} 1 - f^*f \\ \text{invertible} \end{array}}$

Proposition:

$a \geq 1$ implies a is invertible

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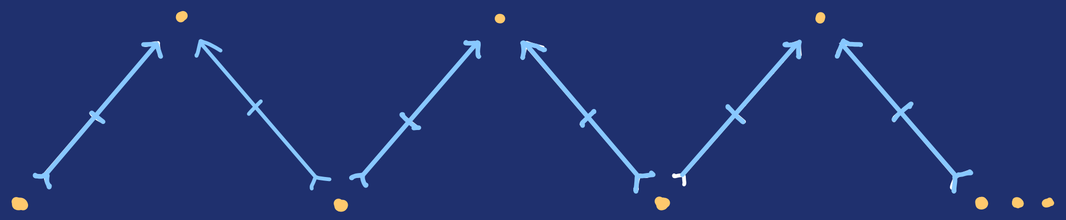
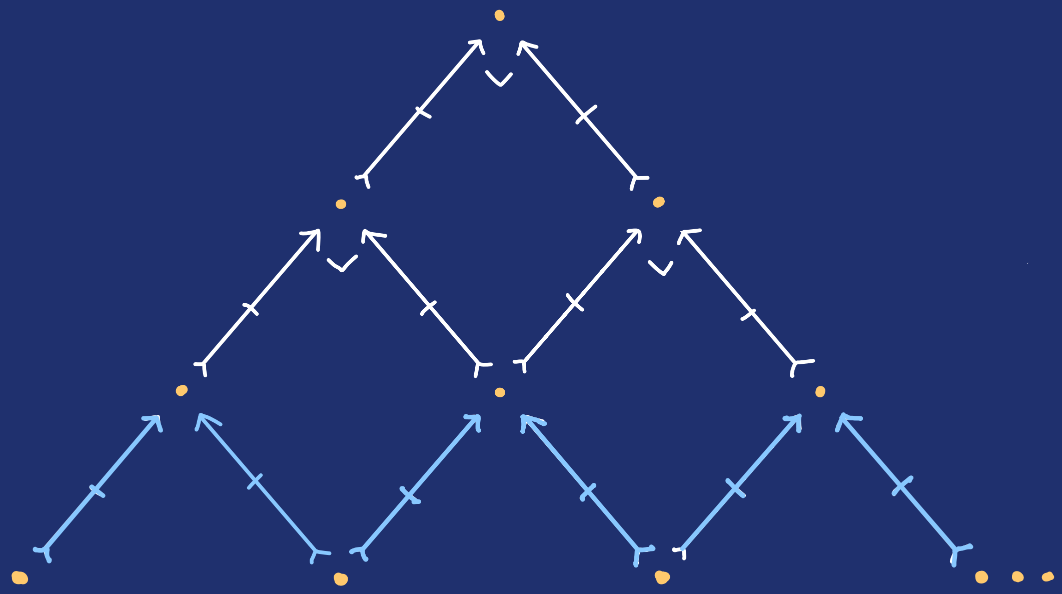
Trick:

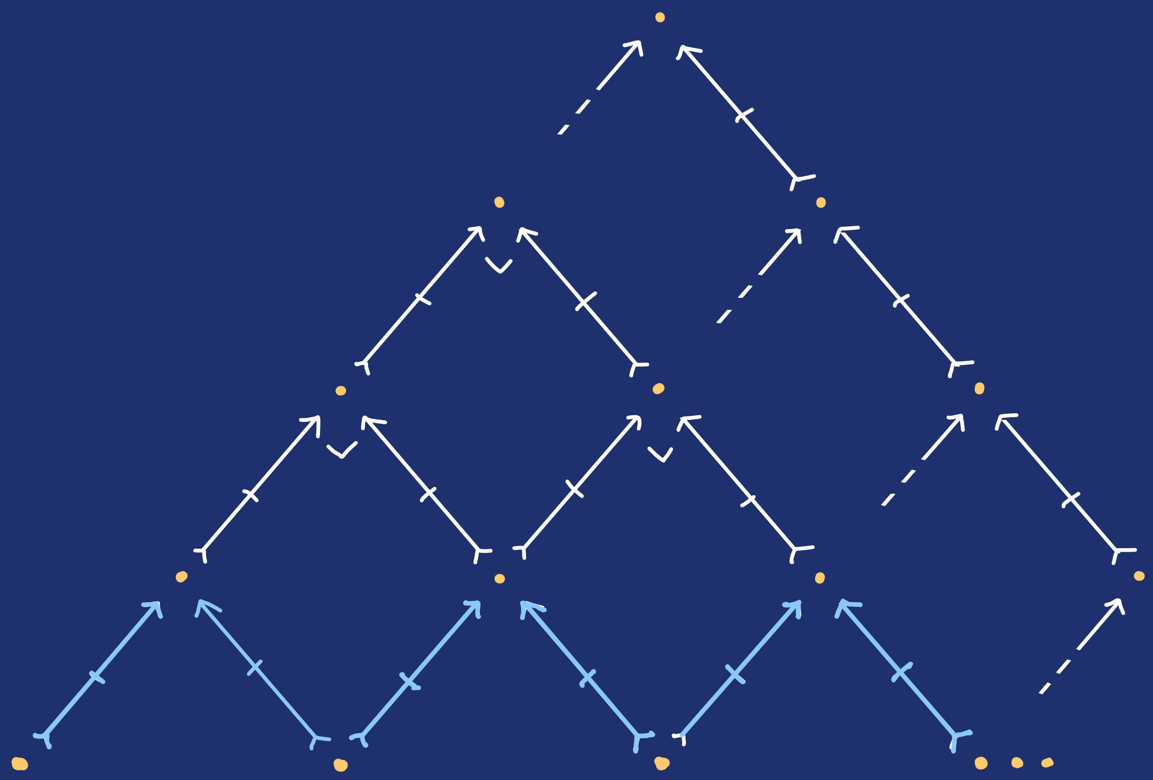
If $f : X \to Y$ is a contraction, then

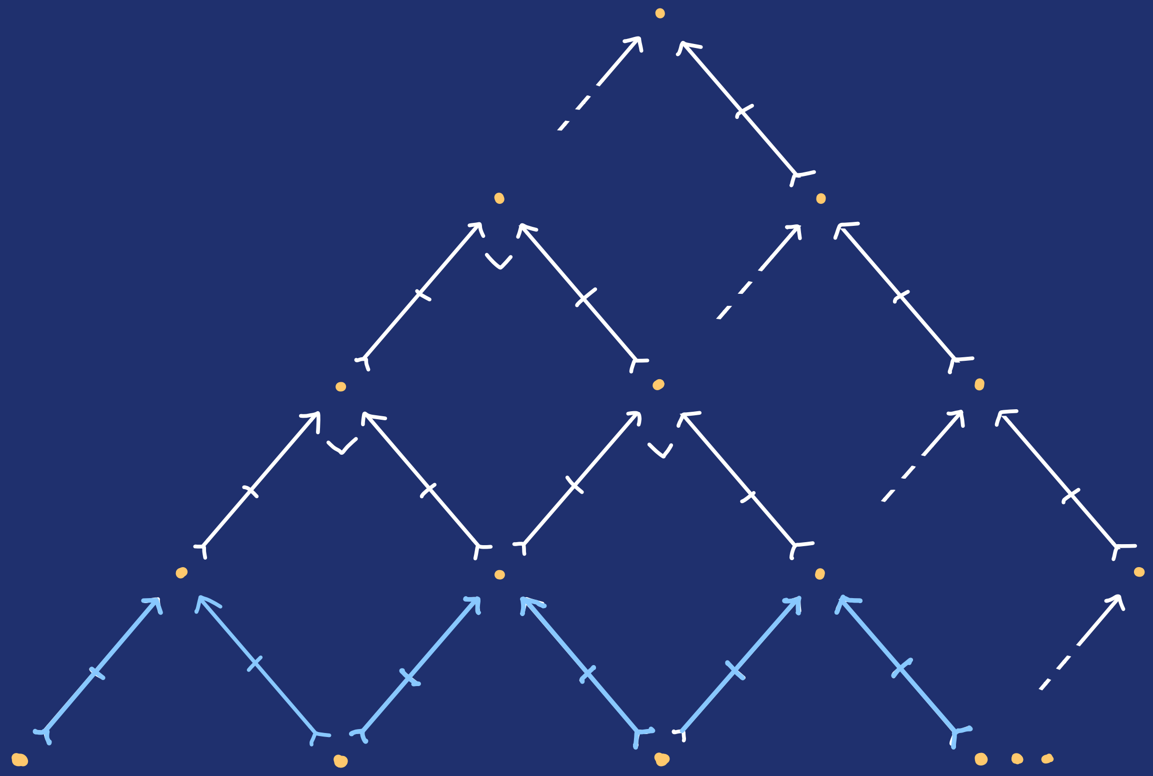
$$(1-4^{-n})f \qquad n = 1, 2, \ldots$$

is a strict contraction.

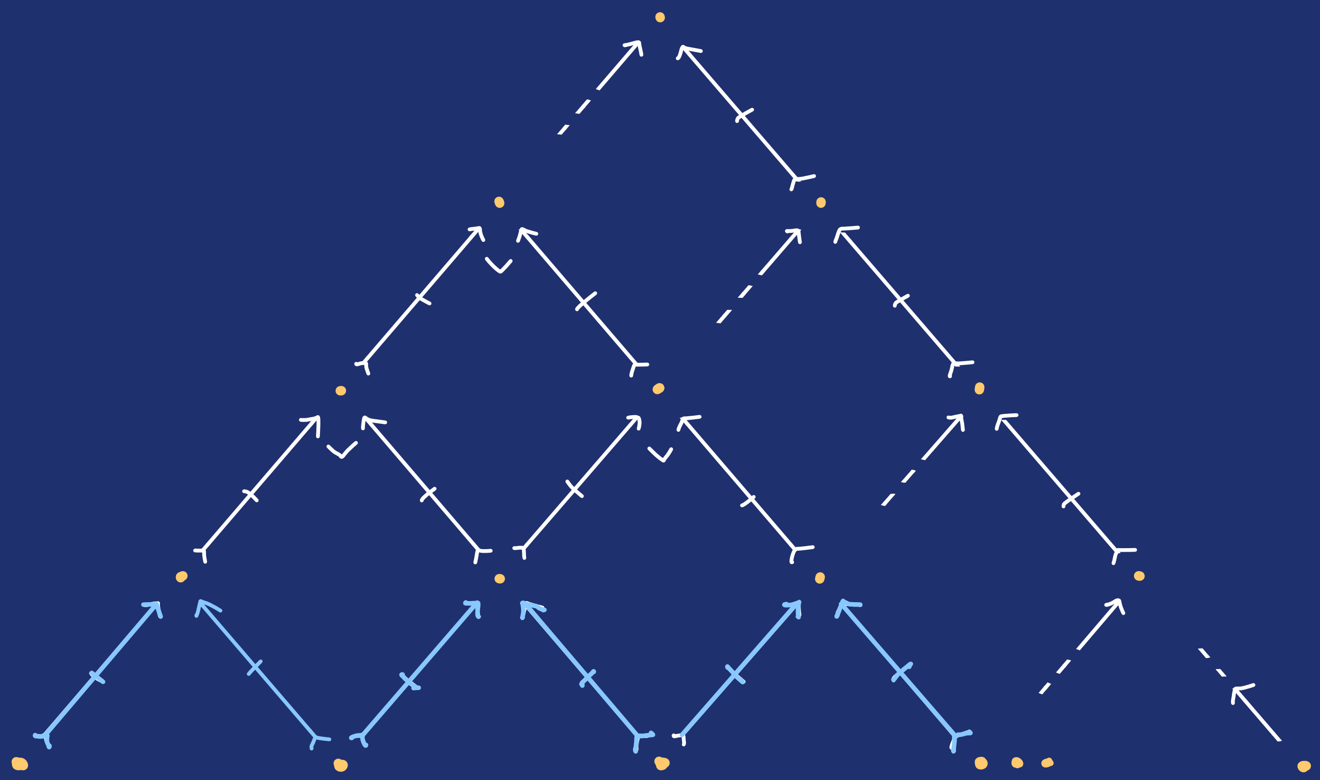








$$C_1/X \xrightarrow[\sim]{(-)^{\perp}} (C_1/X)^{op}$$



$$C_1/X \xrightarrow[\sim]{(-)^{\perp}} (C_1/X)^{op}$$

Preview of next steps

DEFINITION:

An M*-ring is a partially ordered *-ring that is symmetric, monotone complete and orthogonally complete.

$\boxed{a \geq 1 \text{ implies } a \text{ invertible}}$

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DEFINITION:

A Hilbert module over an M*-ring R is an inner product right R-module that is orthogonally complete

M*-rings generalise
monotone complete C*-algebras
in two ways:

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(i) $\|r\|$ could be ∞

eg. Arens algebra

$L^{\omega}[0,1] = \bigcap_{p \geq 1} L^p[0,1]$

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in two ways:

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$L^{\omega}[0,1] = \bigcap_{p \geq 1} L^{p}[0,1]$

(ii) not necessarily complex

PROPOSITION:

If R is complex or commutative then $\mathbf{Hilb}_R$ is an M*-category

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If R is complex or commutative then $\mathbf{Hilb}_R$ is an M*-category

Theorem:

If $\mathbf{C}$ is a complex M*-category with a separator A, then

$$\begin{tikzcd} & \mathbf{Hilb}_{C(A,A)} \arrow[d] \\ \mathbf{C} \arrow[ur, dashed, "F"] \arrow[r, "C(A,-)"'] & \mathbf{Set} \end{tikzcd}$$

where F is unitarily e.s.o, faithful, and full on contractions

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R*-CATEGORIES

THE HILBERT-SPACE ANALOGUE OF ABELIAN CATEGORIES

MATTHEW DI MEGLIO

Abstract. This article introduces R*-categories—an abstraction of categories exhibiting the "algebraic" aspects of the theory of Hilbert spaces. Notably, finite biproducts in R*-categories can be orthogonalised using the Gram–Schmidt process, and generalised notions of positivity and contraction support a variant of Sz.-Nagy's unitary dilation theorem. Underpinning these generalisations is the structure of an involutive identity-on-objects contravariant endofunctor, which encodes adjoints of morphisms. The R*-category axioms are otherwise inspired by those for abelian categories, comprising a few simple properties of products and kernels. Additivity is not assumed, but nevertheless follows. In fact, the similarity with abelian categories runs deeper—R*-categories are quasi-abelian and thus homological. Examples include the category of unitary representations of a group, the category of finite-dimensional inner product modules over a partially ordered division ring, and the category of self-dual Hilbert modules over a W*-algebra.

https://mdimeglio.github.io

If R is an M^*-ring, then

(i) R is uniquely an $\mathbb{R}$-algebra,

(ii) The extended norm $\|-\|: R \to [0, \infty]$ defined by

$$\|r\| = \inf \{ \lambda \in [0, \infty] \mid r^* r \leqslant \lambda^2 \}$$

is complete,

(iii) R is a Baer $*$-ring,

(iv) For all $a \geq 0$, exists unique $b \geq 0$ such that $b^2 = a$

If R is an M^*-ring, then

(v) If R is a division ring, then $R \cong \mathbb{R}, \mathbb{C},$ or $\mathbb{H}$

(vi) If R is commutative, exists projection p such that

- Rp is a complex M^*-algebra
- $R(1-p)$ is a real M^*-algebra with trivial involution