Dagger Categories of Relations

The Return of the Regular-ish Category

Matthew Di Meglio

Joint work with Chris Heunen, Paolo Perrone and Dario Stein

Australian Category Seminar

February 2025

Bijection between contractions and relations in category of coisometries

Linear maps $f: X \to Y$ between Hilbert spaces such that $\|fx\| \leq \|x\|$ for all $x \in X$

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Isomorphism classes of jointly monic spans

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Isomorphism classes of jointly monic spans

Adjointable maps $f : X \to Y$ such that $ff^{\dagger} = 1$ (orthogonal projections onto closed subspaces)

But category of coisometries

is not regular

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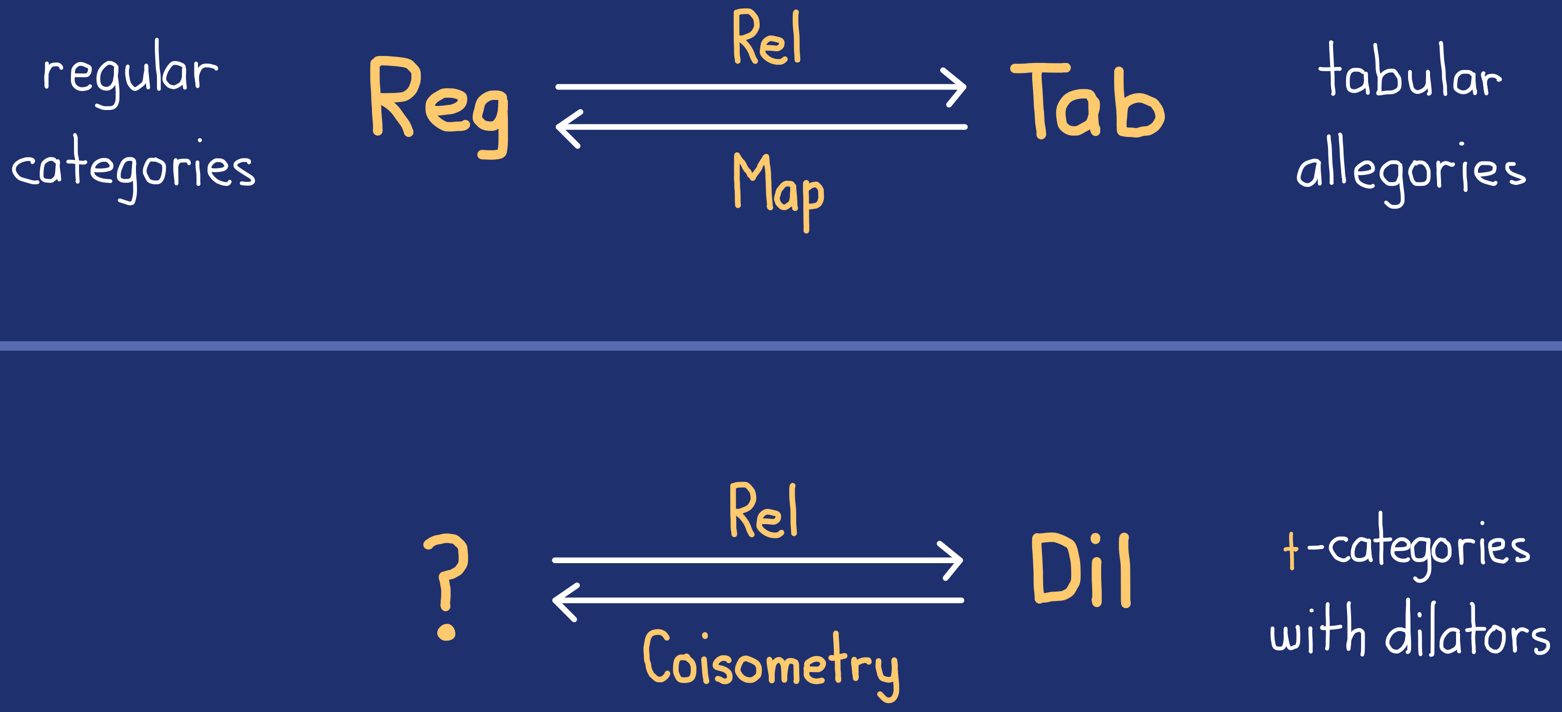
is not regular

It does not have pullbacks

regular categories **Reg** $\underset{\text{Map}}{\overset{\text{Rel}}{\rightleftarrows}}$ **Tab** tabular allegories


regular categories
Reg
Rel
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tabular allegories
?
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Coisometry
Dil
†-categories with dilators

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A **†-category** is a category with $f^\dagger : Y \to X$ for each $f : X \to Y$ such that

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Hilbert spaces and contractions

$$\langle y | fx \rangle = \langle f^\dagger y | x \rangle$$

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The **Bayesian inverse** $f^\dagger$ of f is given by

$$\mathbb{P}_f(y|x)\,\mathbb{P}_X(x) = \mathbb{P}_{f^\dagger}(x|y)\,\mathbb{P}_Y(y)$$

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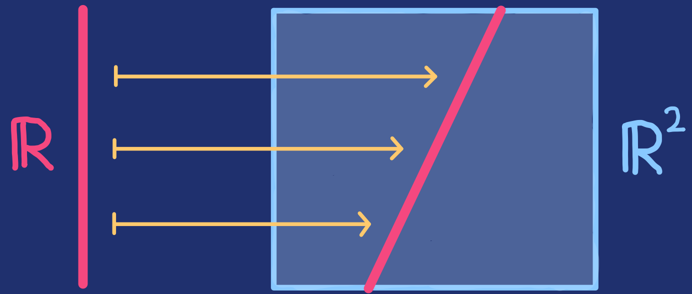
Isometries in **Con** are inclusions of closed subspaces.



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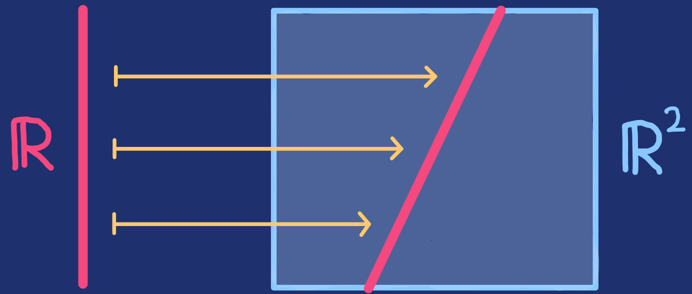
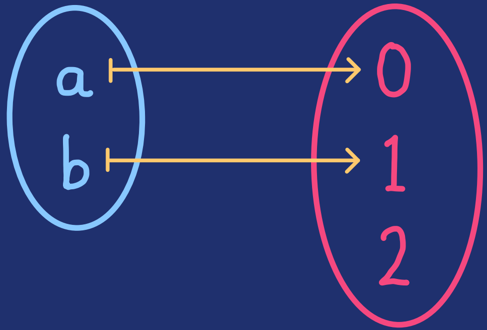


Isometries in **PInj** are total.



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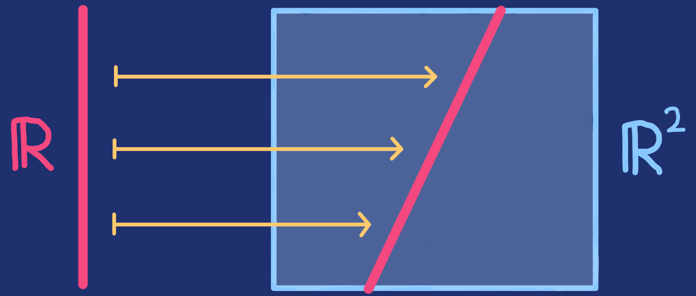

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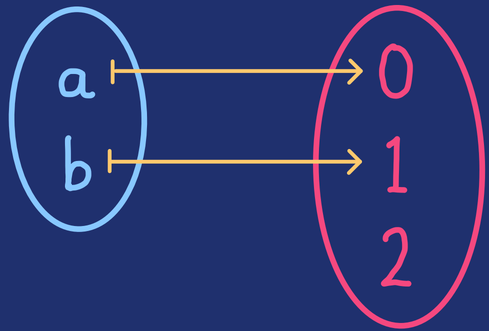

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$$\mathbb{P}_f(y|x) \in \{0, 1\}$$

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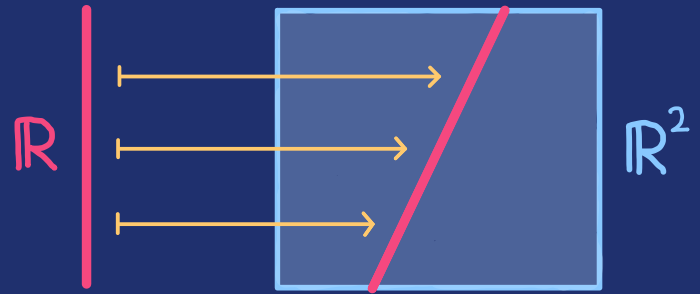

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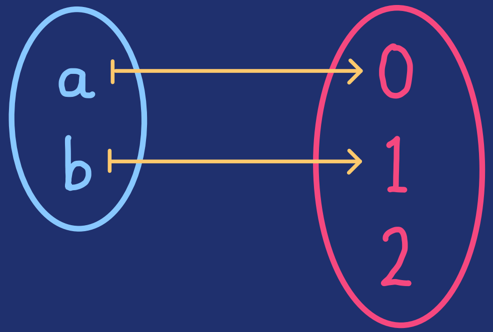

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Write f also for the underlying function

$$\mathbb{P}_f(y|x) = \begin{cases} 1 & \text{if } y = fx \\ 0 & \text{otherwise} \end{cases}$$

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A dilation of $f : X \to Y$ is a span

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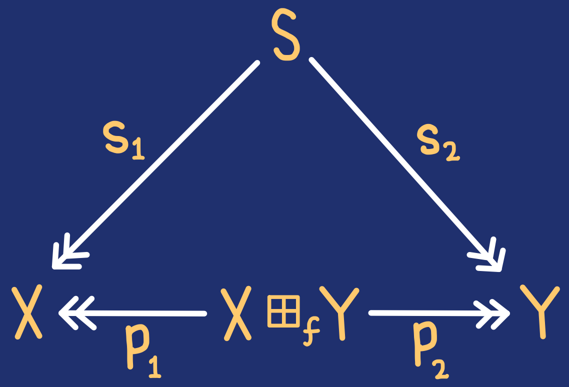

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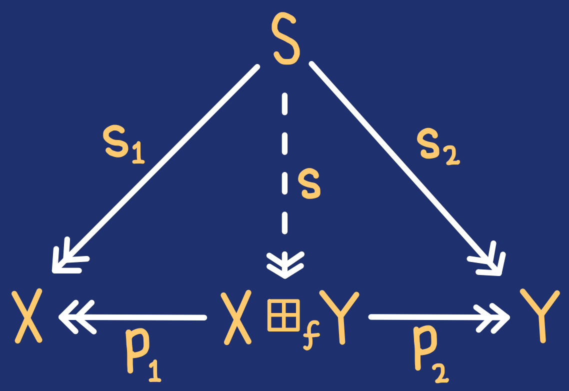

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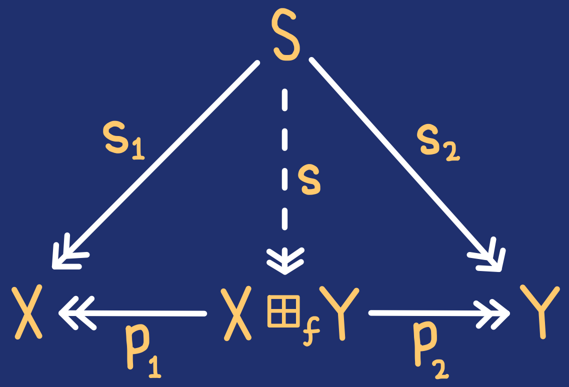

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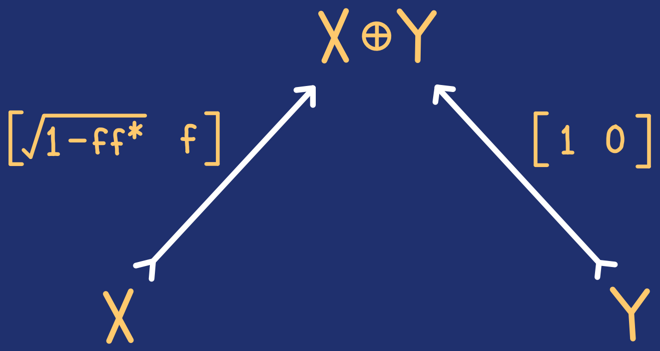

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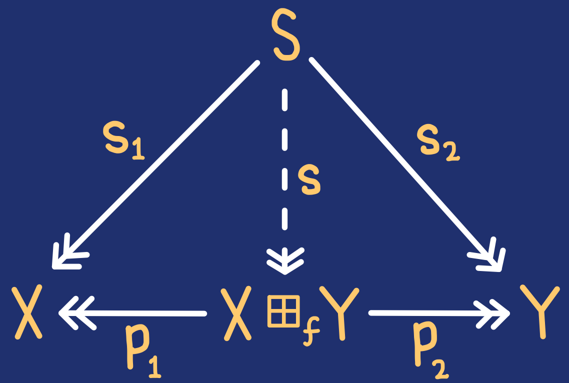

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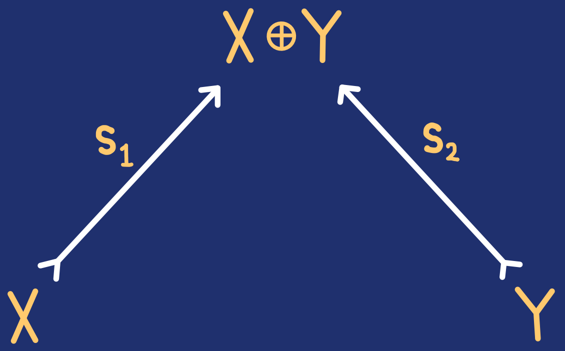

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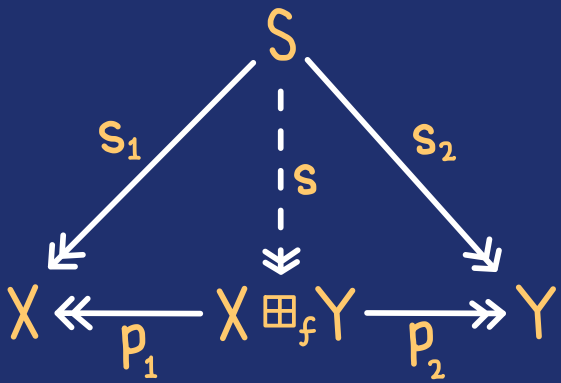

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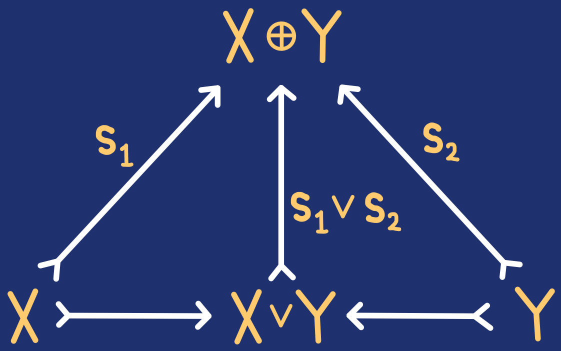

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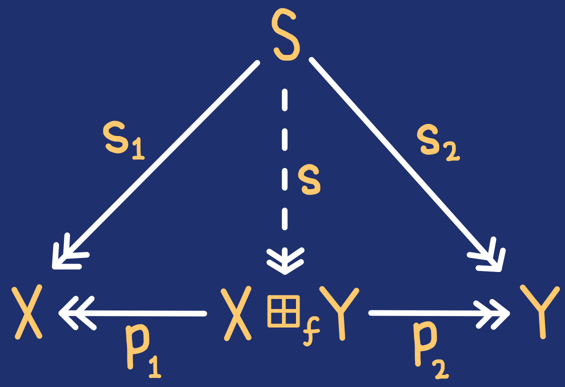

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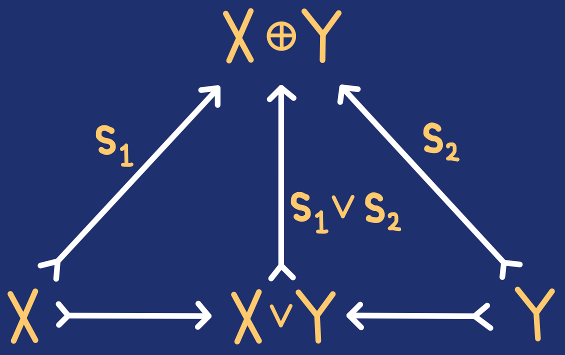

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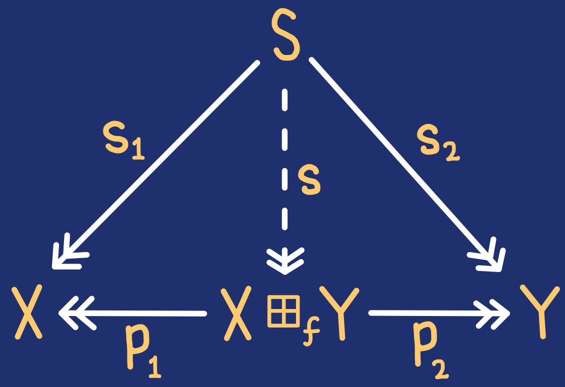

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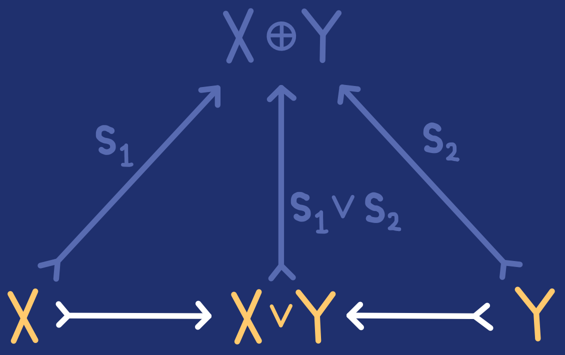

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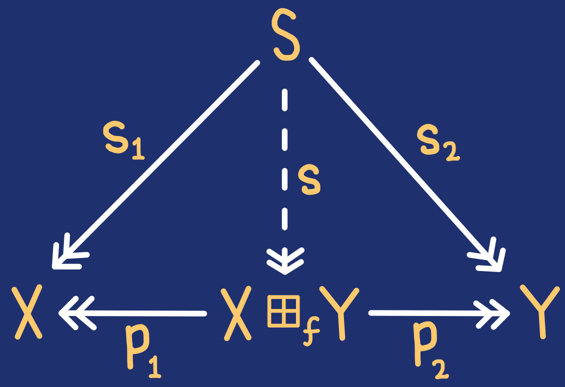

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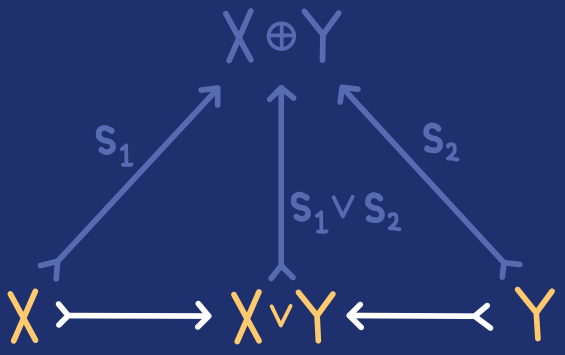

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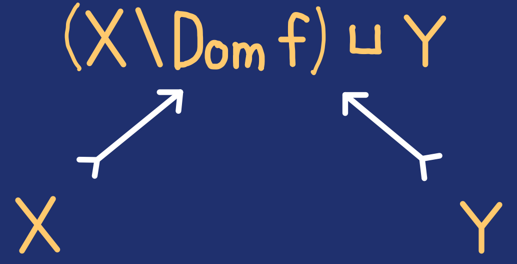

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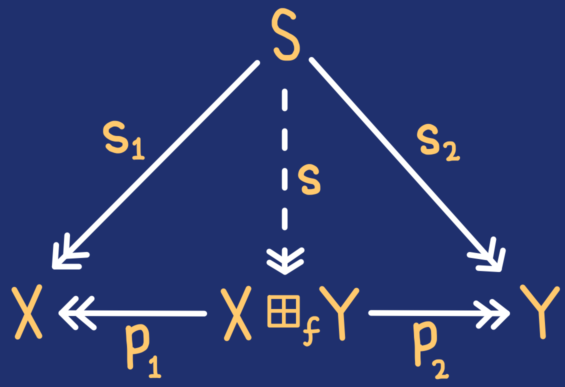

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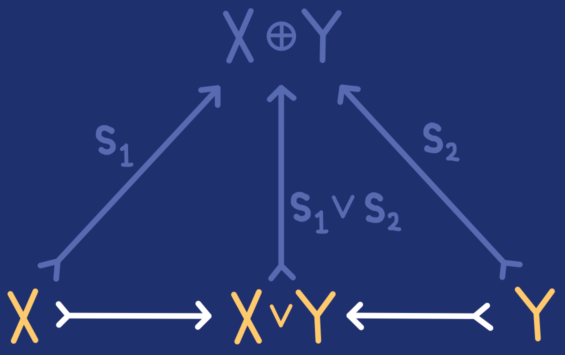

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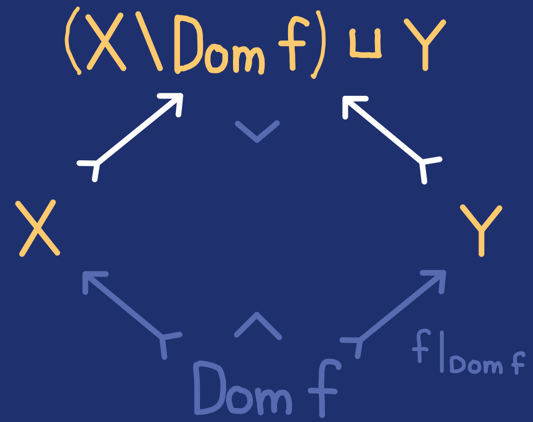

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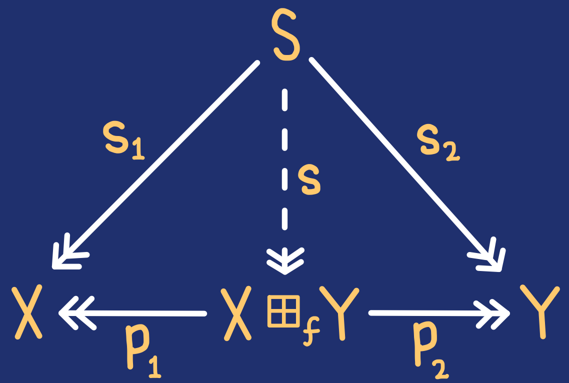

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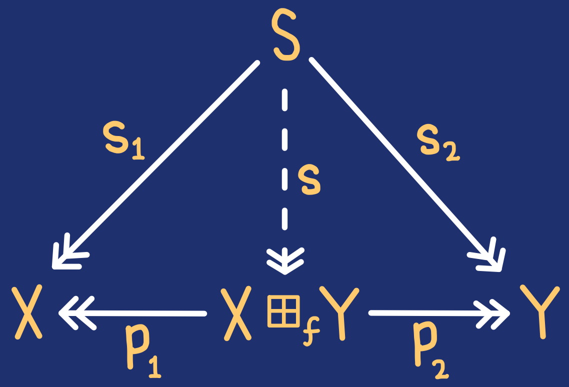

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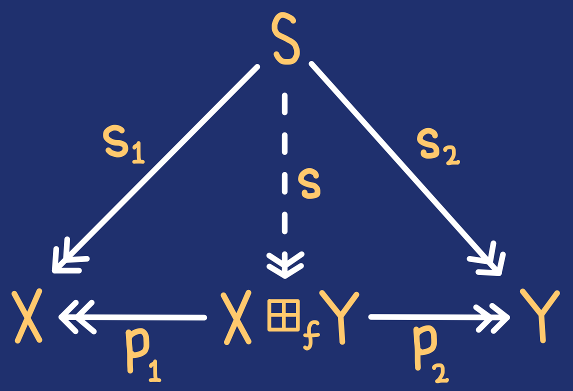

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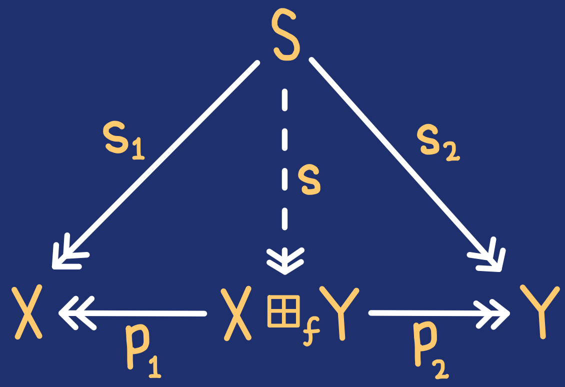

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$$p_1(x, y) = x \qquad p_2(x, y) = y$$

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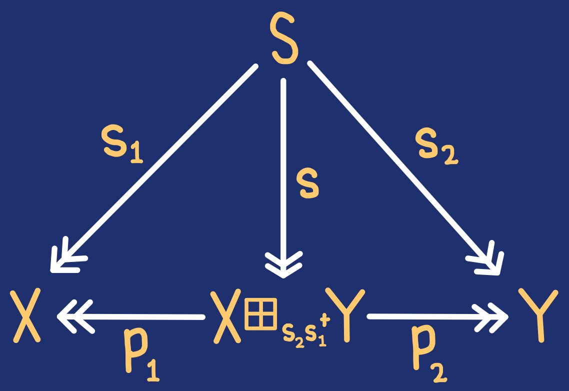

factorisation of spans

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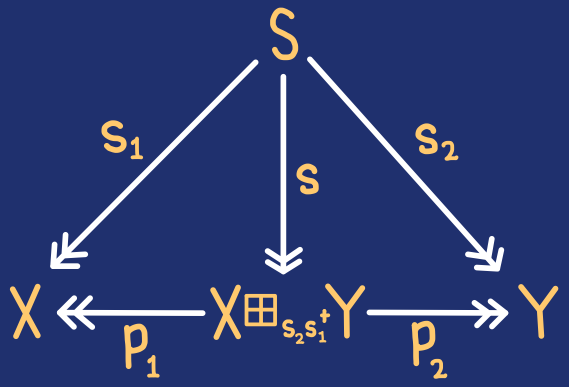

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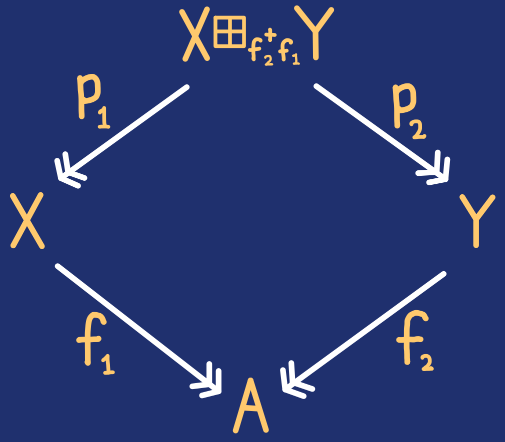

"pullbacks"

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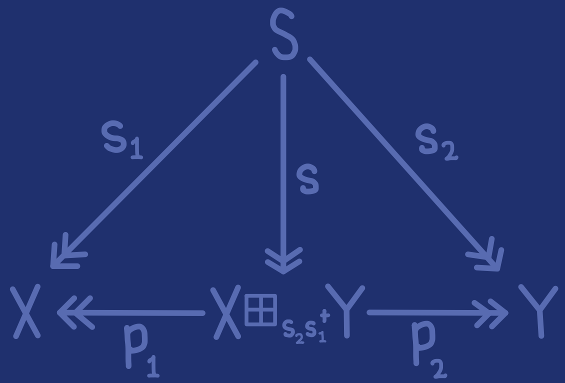

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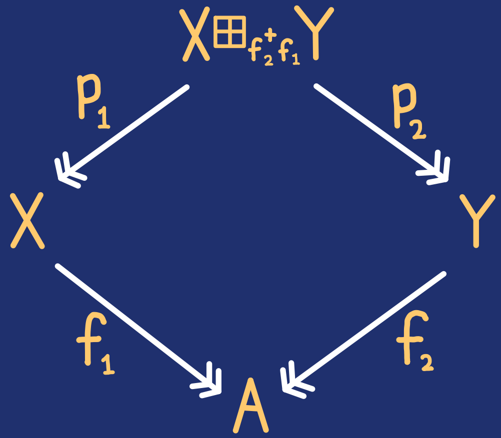

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implies
$$\begin{array}{ccc} \cdot & \xrightarrow{ba} & \cdot \\ {\scriptstyle f}\downarrow & \perp & \downarrow{\scriptstyle h} \\ \cdot & \xrightarrow[ts]{} & \cdot \end{array}$$

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An independence category is a category equipped with a predicate $\perp$ on commuting squares such that

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(ii)

$$\begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle f}\downarrow & \perp & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[1]{} & \cdot \end{array}$$

Inspired by Simpson "Equivalence and Independence in Atomic Sheaf Logic"

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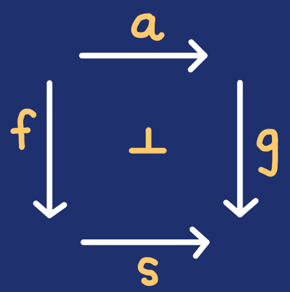
LEMMA

If **C** is a †-category, then **Coisometry(C)** is an independence category where



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$af^{\dagger} = g^{\dagger}s$

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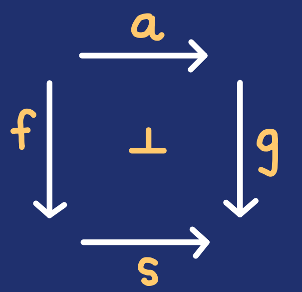

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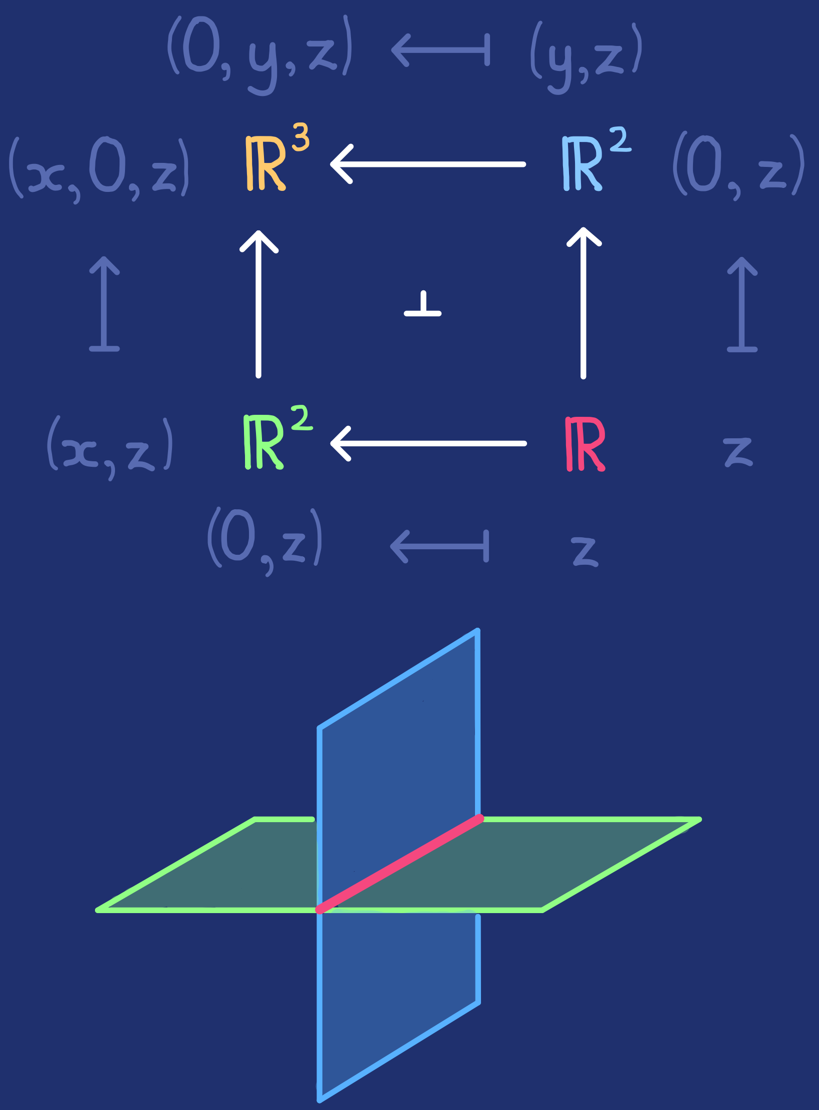

EXAMPLES

In **Isometry(Con)**,

⊥ captures relative orthogonality.

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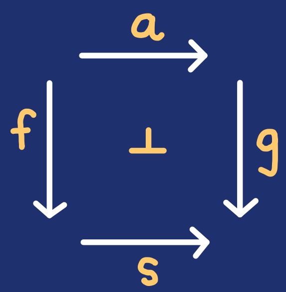
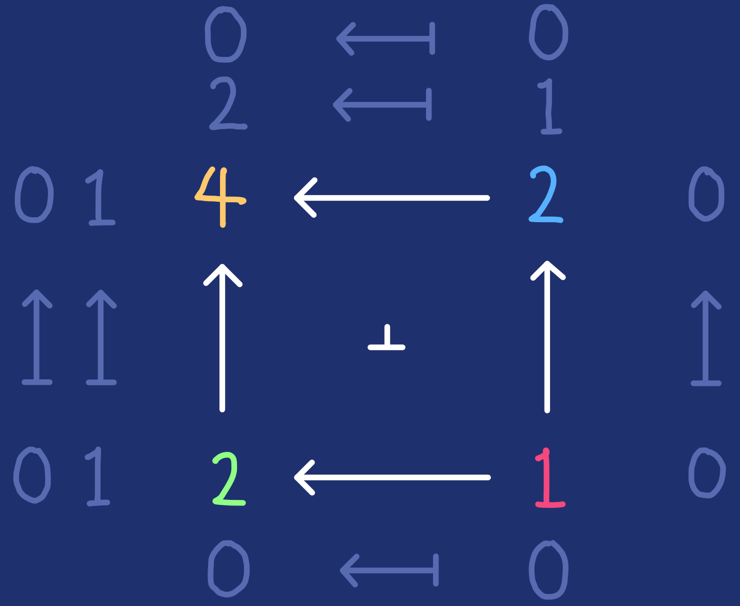


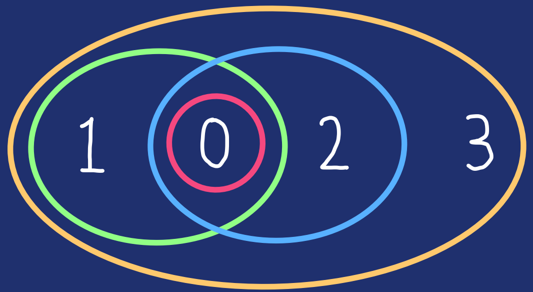
if $\begin{aligned} ga &= sf \\ af^{\dagger} &= g^{\dagger}s \end{aligned}$

EXAMPLES

In **Isometry(PInj)**,

⊥ captures relative disjointness.





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EXAMPLES

In **Coisometry(FinPS)**,

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & \perp & \downarrow{\scriptstyle t} \\ X & \xrightarrow[s]{} & A \end{array}$$

$$\Updownarrow$$

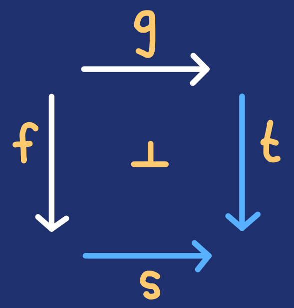
$sf = tg$ and

$$\mathbb{P}_Z(f^{-1}\{x\} \cap g^{-1}\{y\}) = \frac{\mathbb{P}_X(x)\mathbb{P}_Y(y)}{\mathbb{P}_A(a)}$$

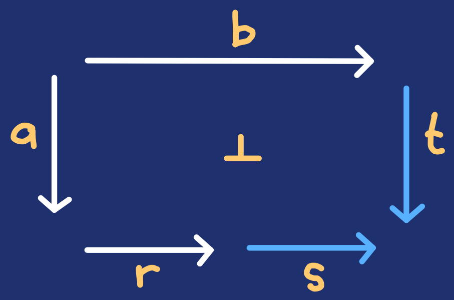
for all $x \in s^{-1}\{a\}$ and $y \in t^{-1}\{a\}$

DEFINITION

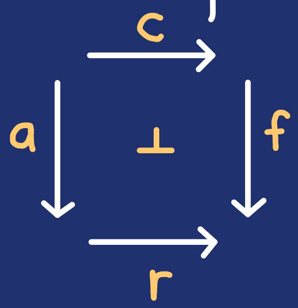
In an independence category, an independent pullback is a square



such that for all



exists unique c such that



and $b = gc$.

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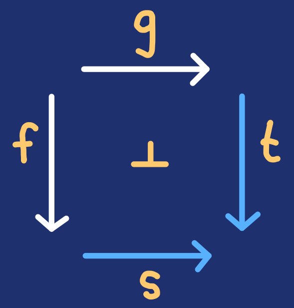

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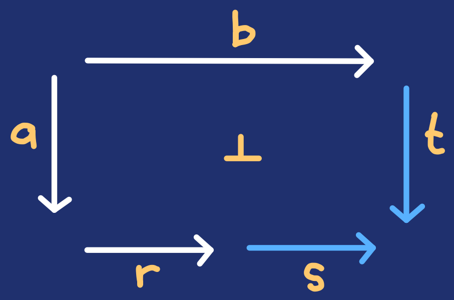

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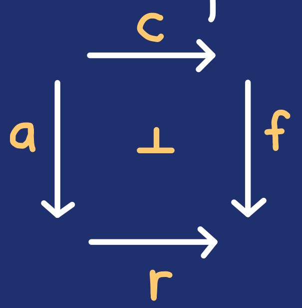

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Weak independent pullbacks are similar, but with $r = 1$.

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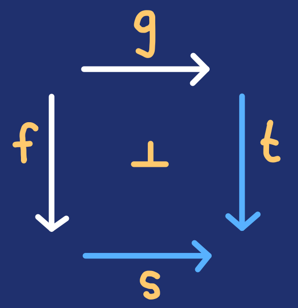

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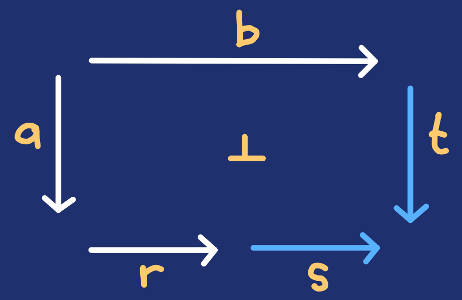

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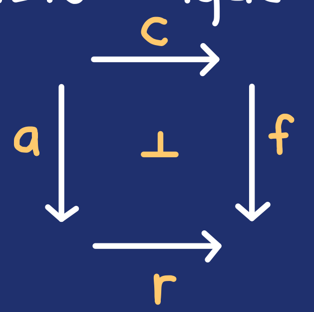

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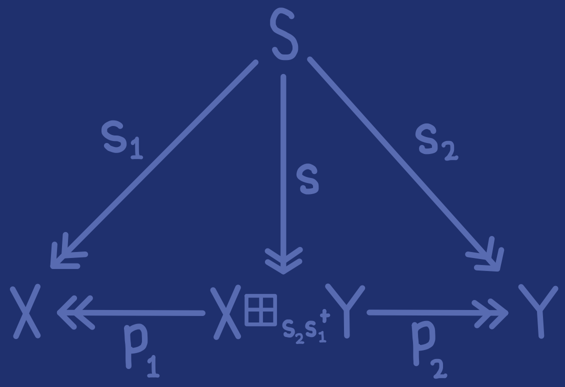
Weak independent pullbacks are similar, but with $r = 1$.

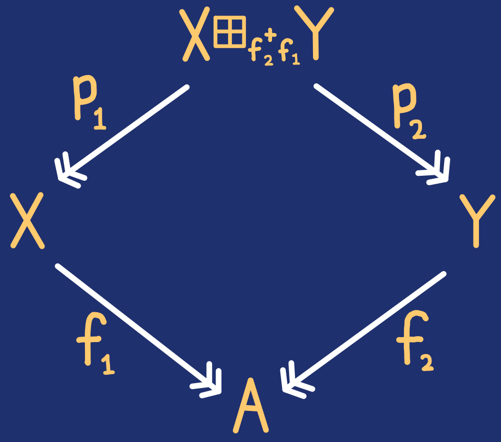
LEMMA

If C is a $\dagger$-category with dilators, then **Coisometry(C)** has weak independent pullbacks.

Remnants of dilators in Coisometry(C)



factorisation of spans



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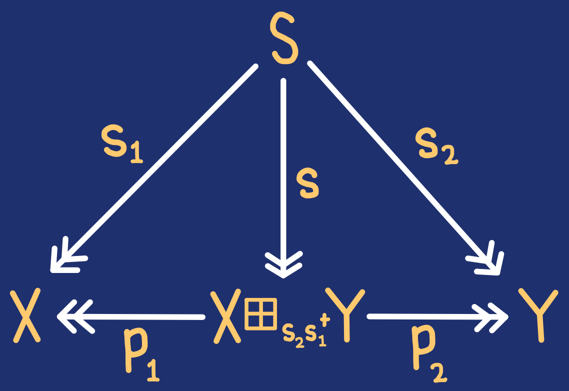

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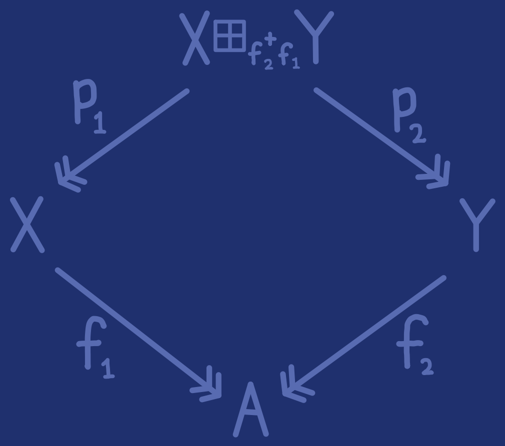

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(Non-standard definition)

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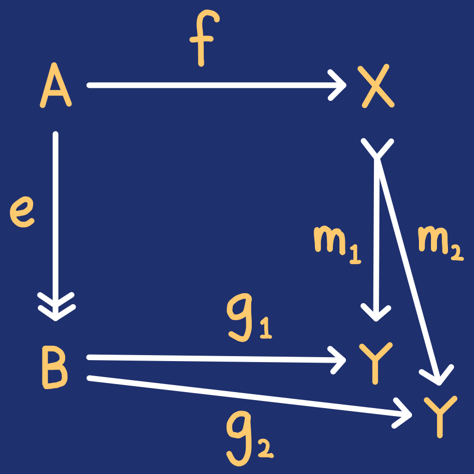

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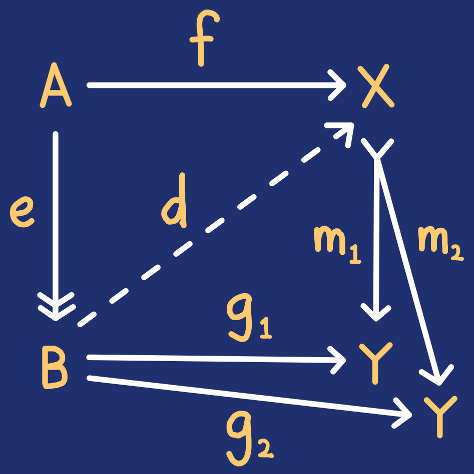

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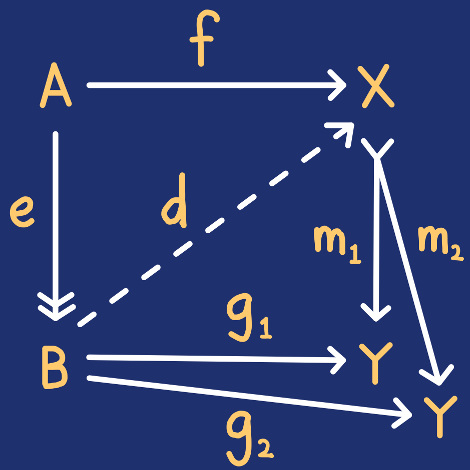

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Let $\mathbf{C}$ be a †-category with dilators. Then

(i) A span in **Coisometry(C)** is jointly monic if and only if it is a dilator in $\mathbf{C}$.

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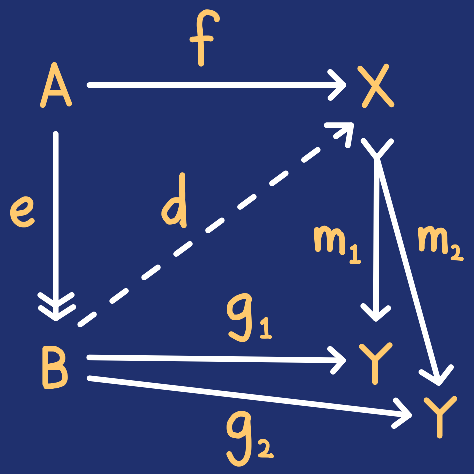

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Let $\mathbf{C}$ be a $\dagger$-category with dilators. Then

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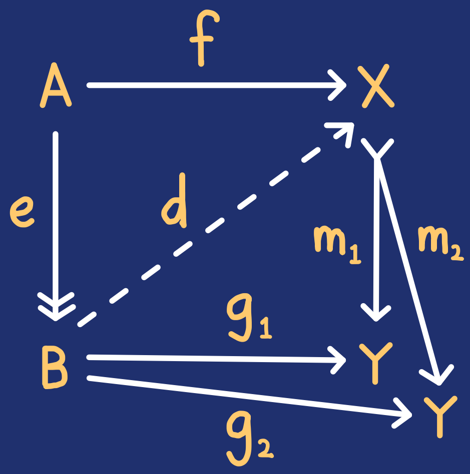

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(iii) Every span in **Coisometry(C)** has a (strong epic, jointly monic) factorisation.

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Let $\mathbf{C}$ be a $\dagger$-category with dilators. Then **Coisometry(C)** is regular-ish.

LEMMA

In a regular-ish independence category every weak independent pullback is an independent pullback.

DEFINITION

Let D be a regular-ish independence category. Define Rel(D) as follows:

- objects are objects of D
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THEOREM

- $Rel(D)$ is a dagger category with dilators.
- $Coisometry(Rel(D)) \cong D$
- $Rel(Coisometry(C)) \cong C$

What is the connection between dilators and tabulators?

Rel is a poset-enriched †-category.

$f \leq g$ implies $f^\dagger \leq g^\dagger$

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Let $f : X \to Y$ be a relation.

$$X \xleftarrow{[1 \;\; p_1 \;\; \bot]} X \sqcup f \sqcup Y \xrightarrow{[\bot \;\; p_2 \;\; 1]} Y$$

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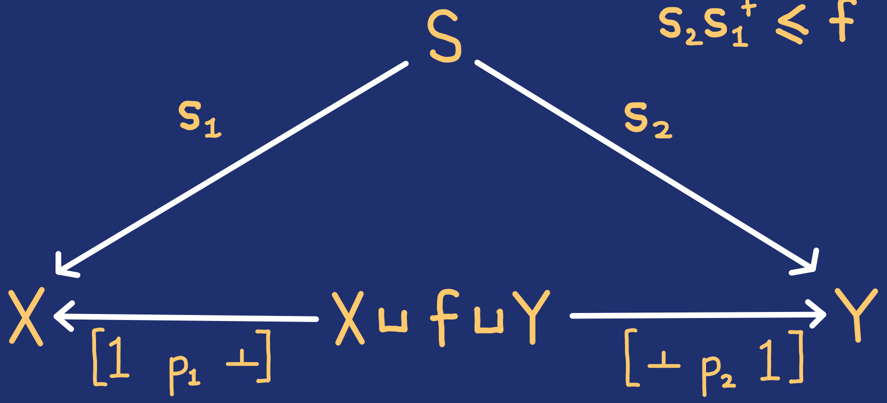

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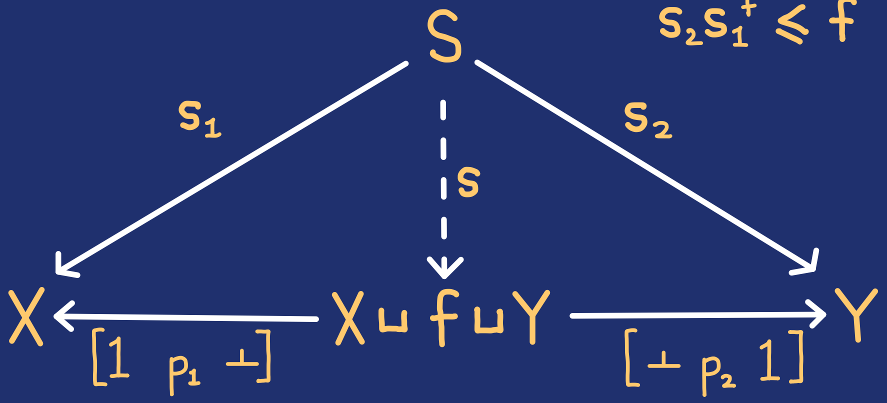

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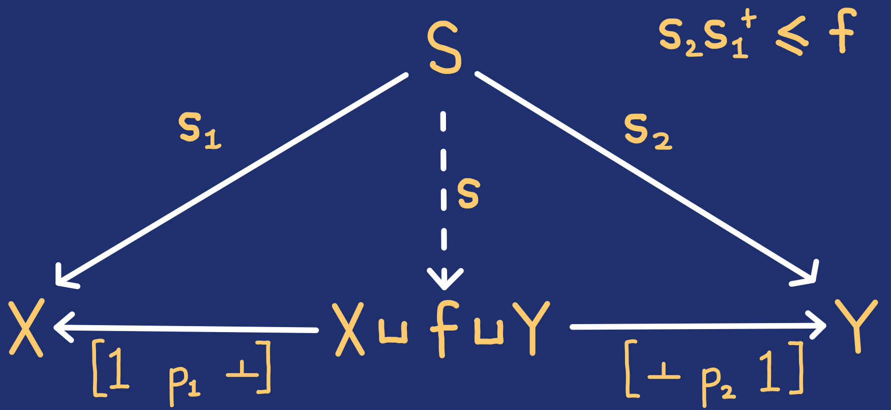

A morphism $f : (X, x) \to (Y, y)$ in $\mathbf{Rel}_*$ is a relation $f : X \to Y$ such that $(x, y) \in f$.

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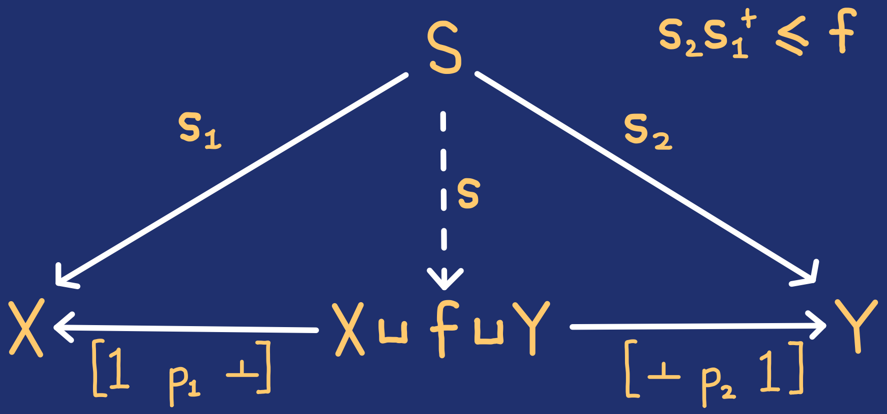

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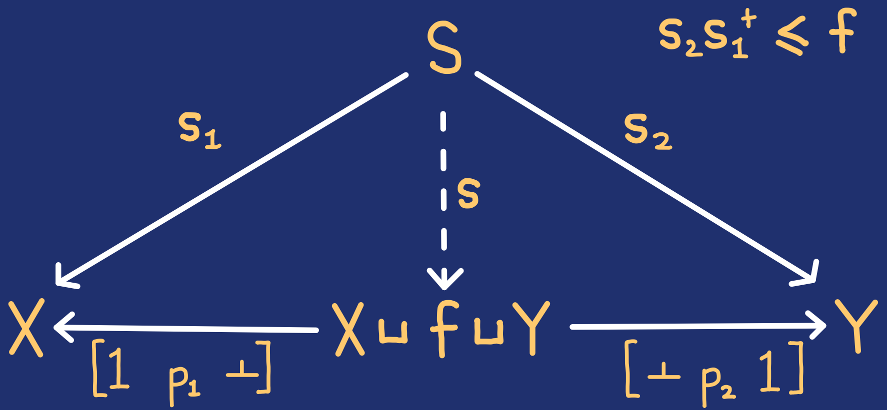

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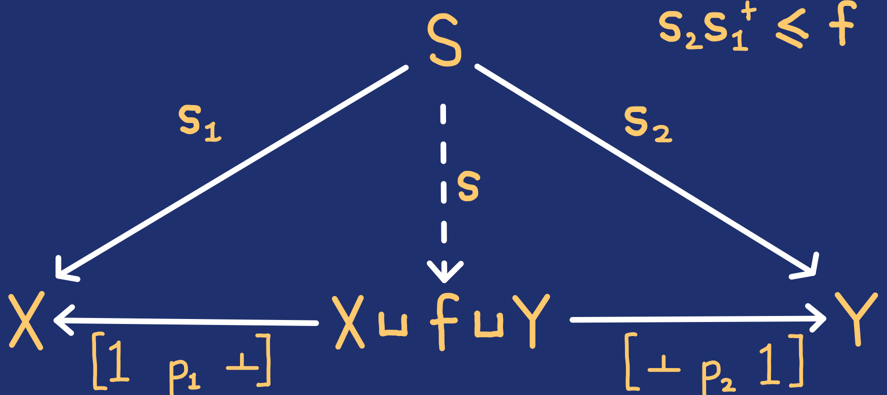

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$$X$$

$$\downarrow$$

$$(X \sqcup \{*\}, *)$$

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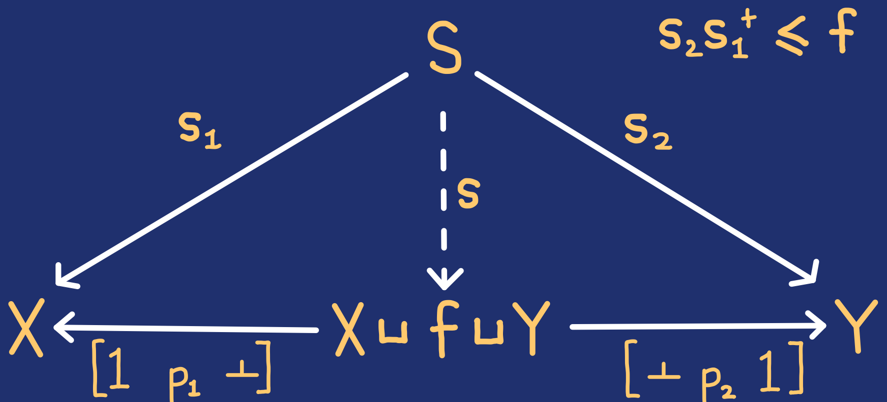

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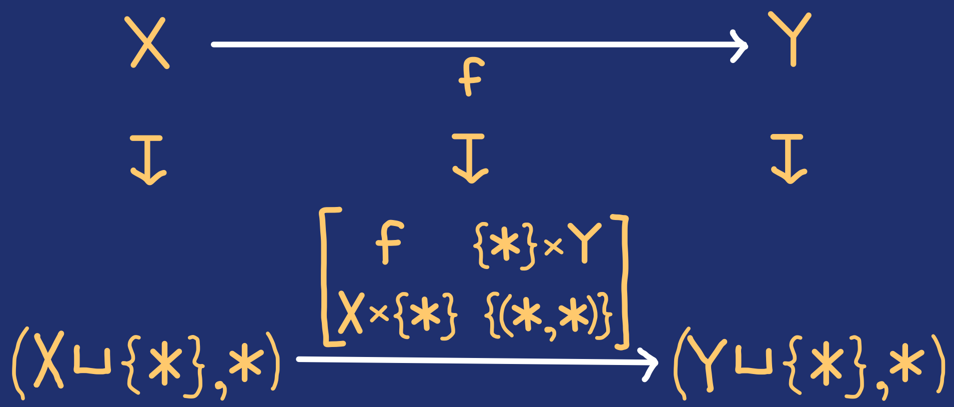

PROJECTS

With Heunen, Perrone and Stein

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With Heunen, Perrone and Stein

Characterise the category of Hilbert spaces and coisometries

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https://mdimeglio.github.io

All coisometries have a quantum interpretation

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Not a $\dagger$-category

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With Heunen, Perrone and Stein

Characterise the category of Hilbert spaces and coisometries

Characterise a category of probability spaces

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PROJECTS

With Heunen, Perrone and Stein

All coisometries have a quantum interpretation

The first characterisation in categorical probability

Characterise the category of Hilbert spaces and coisometries

Characterise a category of probability spaces

Not a †-category

Not Markov categories

m.dimeglio@ed.ac.uk

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