Axioms for the category of finite-dimensional Hilbert spaces and linear contractions

Matthew Di Meglio and Chris Heunen

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THE UNIVERSITY of EDINBURGH

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INTRODUCTION

Examples of categorical axiomatisation

homological algebra	$\rightsquigarrow$	abelian categories
probability theory	$\rightsquigarrow$	Markov categories
differential geometry	$\rightsquigarrow$	tangent categories
logic and set theory	$\rightsquigarrow$	elementary topoi

Features of categorical axiomatisation

- Shift focus from internal structure to relationships between objects
- Uniform treatment of similar kinds of mathematical structures
- Compare different kinds of mathematical structures

Long-term goal:

A similar categorical theory of Hilbert spaces

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Background

Hilbert spaces

- Vector spaces with geometry (encoded in an *inner product*)

$$\|x\| = \sqrt{\langle x|x\rangle} \qquad \cos\theta = \frac{\langle x|y\rangle}{\|x\|\,\|y\|}$$

  and no “gaps” (e.g., $\mathbb{Q}$ has “gaps” whereas $\mathbb{R}$ does not)

- Model the states of a quantum system

- Every n-dimensional (real) Hilbert space is isomorphic to $\mathbb{R}^n$ with

$$\big\langle (x_1, x_2, \ldots, x_n) \big| (y_1, y_2, \ldots, y_n) \big\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

- $\ell_2(\mathbb{N}) = \left\{ (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} \,\middle|\, {x_1}^2 + {x_2}^2 + \cdots < \infty \right\}$ with

$$\big\langle (x_1, x_2, \ldots) \big| (y_1, y_2, \ldots) \big\rangle = x_1 y_1 + x_2 y_2 + \ldots$$

Linear contractions

- Linear maps that decrease lengths

$$\|x\| \geqslant \|Tx\|$$

- Suffice to describe the evolution of pure quantum states (include all unitaries and projections)

- **Con** is the category of Hilbert spaces and linear contractions

- **FCon** is the full subcategory of finite-dimensional Hilbert spaces

Adjoints

- Underpin the categorical treatment of Hilbert spaces

- The *adjoint* of a linear contraction $T\colon X \to Y$ between Hilbert spaces is the unique linear contraction $T^\dagger\colon Y \to X$ such that

$$\langle Tx|y\rangle = \langle x|T^\dagger y\rangle$$

- The matrix representation of $T^\dagger\colon \mathbb{R}^m \to \mathbb{R}^n$ is the transpose of the matrix representation of $T\colon \mathbb{R}^n \to \mathbb{R}^m$

Dagger categories

- A *dagger category* is a category equipped with a choice of $f^\dagger : Y \to X$ for each $f : X \to Y$, such that

$$1^\dagger = 1 \qquad (gf)^\dagger = f^\dagger g^\dagger \qquad (f^\dagger)^\dagger = f$$

- Examples include **Con** and **FCon** where the dagger is the adjoint

Characterisation of Con

Theorem (Heunen, Kornell and van der Schaaf)

A dagger rig category $(\mathbf{D}, \otimes, I, \oplus, O)$ *is equivalent to* **Con** *if and only if*

1. O *is initial,*
2. $i_1 = (I \cong I \oplus O \xrightarrow{1 \oplus 0} I \oplus I)$ *and* $i_2 = (I \cong O \oplus I \xrightarrow{0 \oplus 1} I \oplus I)$ *are jointly epic,*

⋮

8. *every dagger monomorphism is a kernel,*
9. *every directed diagram has a colimit.*

Goal for today:

A similar characterisation of FCon

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Issues adapting the characterisation of Con

Need to understand how the equivalence is constructed

Scalars and vectors from contractions

$$\begin{array}{ccc} f & \mapsto & f(1) \\ \mathsf{Con}(\mathbb{R}, X) & \cong & \{x \in X \mid \|x\| \leqslant 1\} \\ (a \mapsto a \cdot x) & \leftarrow\!\shortmid & x \end{array}$$

$$\mathsf{Con}(\mathbb{R}, \mathbb{R}) \cong \{a \in \mathbb{R} \mid |a| \leqslant 1\}$$

$$X = \left\{\tfrac{1}{a} \cdot x \,\middle|\, x \in X, \|x\| \leqslant 1, a \in \mathbb{R}, |a| \leqslant 1, a \neq 0\right\}$$

The scalar localisation

- The object I corresponds to the 1-dimensional space $\mathbb{R}$
- Construct $\mathbf{C}$ from $\mathbf{D}$ by "formally inverting" the elements of $\mathbf{D}(I, I)\backslash\{0\}$

$$\mathbf{C}(I, X) = \big(\mathbf{D}(I, X) \times \mathbf{D}(I, I)\backslash\{0\}\big)/\sim$$

$$(x, a) \sim (y, b) \iff xb = ya$$

- Write $\frac{x}{a}$ for the equivalence class of (x, a)
- $\mathbf{C}(I, I)$ is an involutive field
- $\mathbf{C}(I, X)$ is a ***Hermitian space*** over $\mathbf{C}(I, I)$ with $\langle x|y\rangle = x^{\dagger}y$.

Solèr's theorem

- Gives sufficient conditions for a Hermitian space over an involutive field to be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$
- To show that $\mathbf{C}(I, I)$ is $\mathbb{R}$ or $\mathbb{C}$, suffices to construct an object X such that $\mathbf{C}(I, X)$ satisfies these conditions
- $\mathbf{C}(I, X)$ must be infinite dimensional
- Known proofs of Solèr's theorem are not conceptual

Subgoal:

A conceptual proof that the field of scalars of C is $\mathbb{R}$ or $\mathbb{C}$ that does not use infinite dimensionality

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Limits of sequences from limits of diagrams

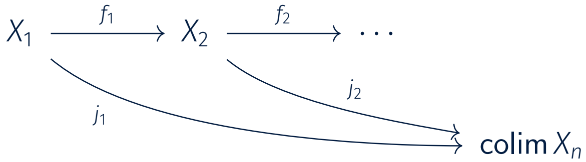
Sequential colimits of contractions

Con has *sequential colimits*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

Sequential colimits of contractions

Con has *sequential colimits*



For each $x \in X_1$,

$$\|x\| \geqslant \|f_1 x\| \geqslant \|f_2 f_1 x\| \geqslant \dots$$

$$\|j_1 x\| = \inf_{n \in \mathbb{N}} \|f_n \dots f_2 f_1 x\| = \lim_{n \to \infty} \|f_n \dots f_2 f_1 x\|$$

BIG IDEA:

Turn these observations about Con into definitions about D and C.

Positivity and order

Suppose that **D** satisfies the axioms for **Con**, and let **C** be the scalar localisation of **D**

$$P = \left\{ a \in \mathbf{C}(I, I) \,\middle|\, a = x^{\dagger}x \text{ for some } X \text{ and some } x \in \mathbf{C}(I, X) \right\}$$

$$x^{\dagger}x \geqslant y^{\dagger}y \qquad \Longleftrightarrow \qquad y = fx \text{ for some } f \in \mathbf{D}(X, Y)$$

Infima from sequential colimits

$$x_1^\dagger x_1 \geqslant x_2^\dagger x_2 \geqslant \cdots$$

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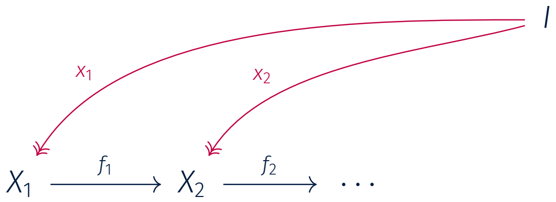

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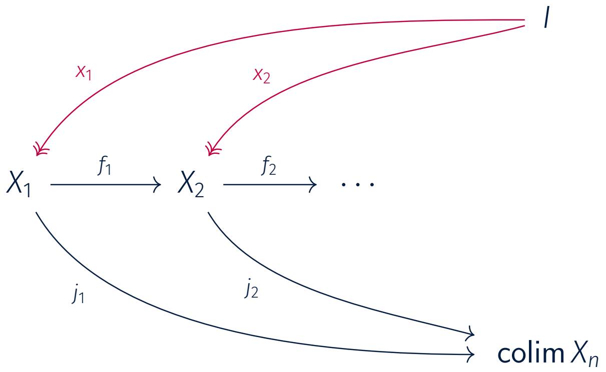

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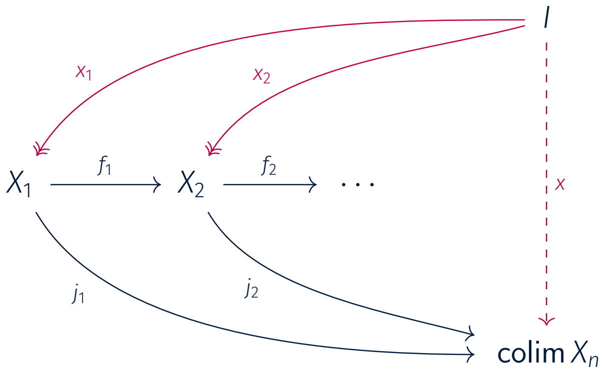
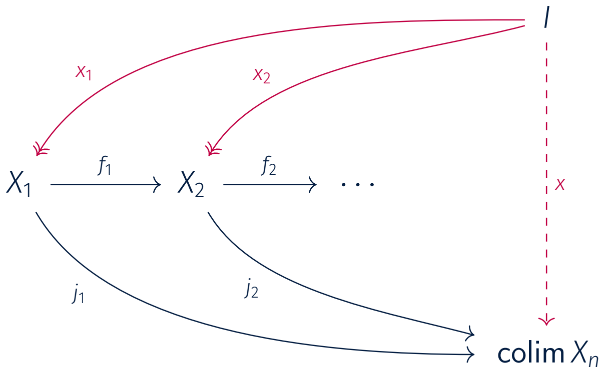


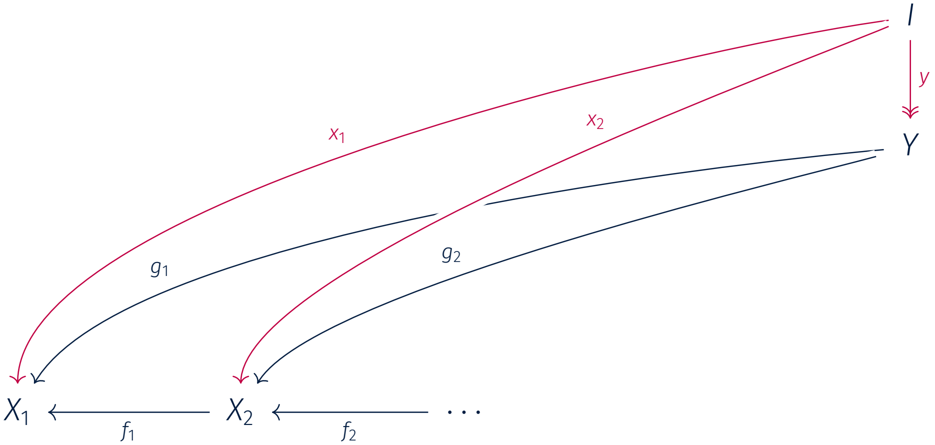
Infima from sequential colimits

$$x_1^\dagger x_1 \quad \geqslant \quad x_2^\dagger x_2 \quad \geqslant \quad \cdots \quad \geqslant \quad x^\dagger x = \inf_{n\in\mathbb{N}} x_n^\dagger x_n$$



Suprema from cosequential limits

$$x_1{}^\dagger x_1 \quad \leqslant \quad x_2{}^\dagger x_2 \quad \leqslant \quad \cdots \qquad\qquad \leqslant \quad y^\dagger y$$



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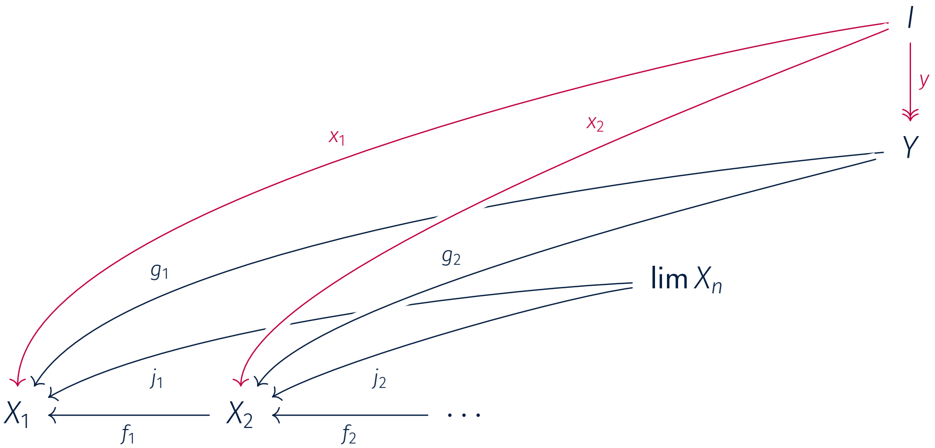

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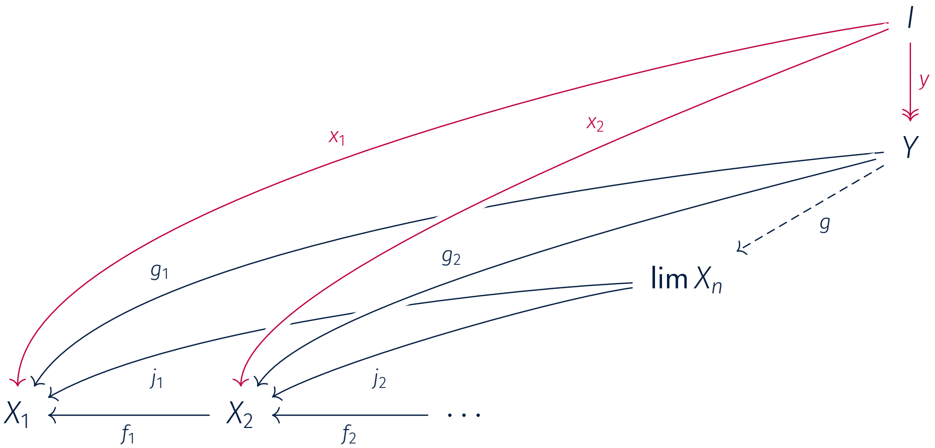

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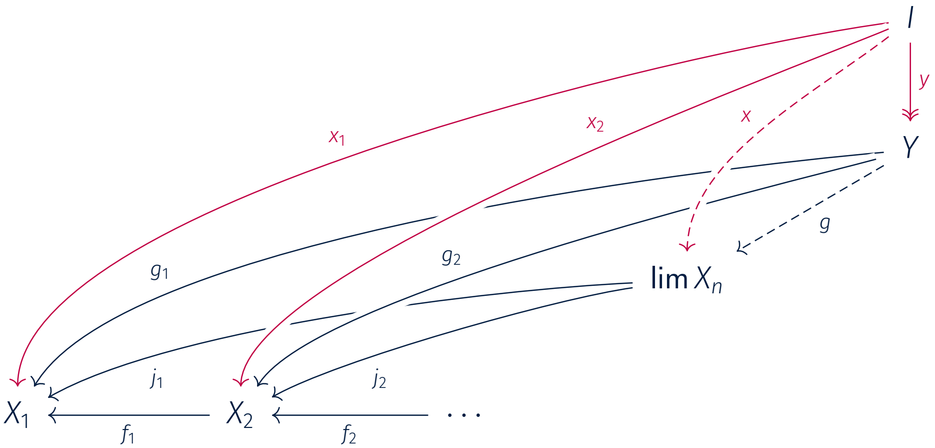

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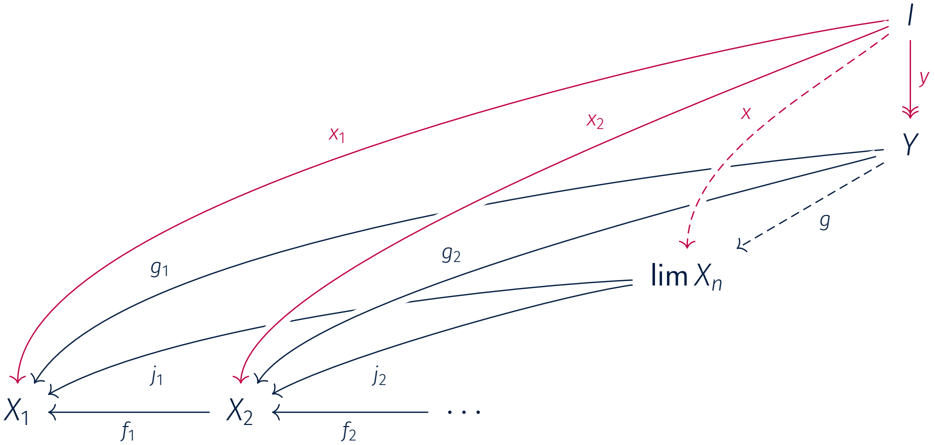
Suprema from cosequential limits



$$x_1^\dagger x_1 \leqslant x_2^\dagger x_2 \leqslant \cdots \leqslant x^\dagger x = \sup_{n\in\mathbb{N}} x_n^\dagger x_n \leqslant y^\dagger y$$

Identifying the real or complex numbers

P is a partially ordered ***strict semifield*** (field without negatives) that has suprema (or infima) of bounded increasing (or decreasing) sequences

Theorem (DeMarr, 1967)

*Every partially ordered **field** that has suprema of bounded increasing sequences is isomorphic to* $\mathbb{R}$

The challenging part of our work was bridging the gap

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Finite dimensionality

Characterisation of Con

Theorem (Heunen, Kornell and van der Schaaf)

A dagger rig category $(\mathbf{D}, \otimes, I, \oplus, O)$ *is equivalent to* **Con** *if and only if*

1. *O is initial,*

   ⋮

8. *every dagger monomorphism is a kernel,*
9. *every directed diagram has a colimit.*

$\ell_2(\mathbb{N})$ is the colimit in **Con** of the sequential diagram

$$\mathbb{R} \xrightarrow{x_1 \mapsto (x_1, 0)} \mathbb{R}^2 \xrightarrow{(x_1, x_2) \mapsto (x_1, x_2, 0)} \mathbb{R}^3 \xrightarrow{(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0)} \cdots$$

Bounded directed diagrams

- Idea: monomorphisms of vector spaces are increasing in dimension
- A diagram is called ***bounded*** if it admits a cocone of monomorphisms
- **FCon** has colimits of bounded sequential diagrams
- Can adapt the construction of suprema to use only these colimits

Dagger finiteness

- A set S is finite if and only if all injective functions $S \to S$ are bijective
- A morphism $f : X \to Y$ is ***dagger monic*** if $f^\dagger f = 1$
- An object X is ***dagger finite*** if all dagger monomorphisms $X \to X$ are isomorphisms
- An object in **Con** is dagger finite if and only if it is finite dimensional

Characterisation of Con

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$\vdots$

8. *every dagger monomorphism is a kernel,*
9. ***every directed diagram has a colimit.***

Characterisation of FCon

Theorem (Di Meglio and Heunen)

A dagger rig category $(\mathbf{D}, \otimes, I, \oplus, O)$ *is equivalent to* **FCon** *if and only if*

1. O *is initial,*
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$\vdots$

8. *every dagger monomorphism is a kernel,*
9. ***every bounded sequential diagram has a colimit,***
10. ***every object is dagger finite.***

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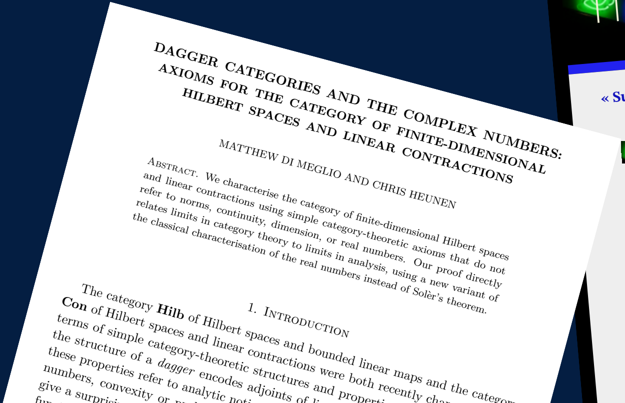
Conclusion

Related and future work

- Rational dagger categories (similar to abelian categories)
- Real dagger categories and axioms for **Hilb** and **FHilb** over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (no monoidal product required)
- Axioms for the category of self-dual Hilbert modules over a monotone complete C*-algebra

Contact me at m.dimeglio@ed.ac.uk



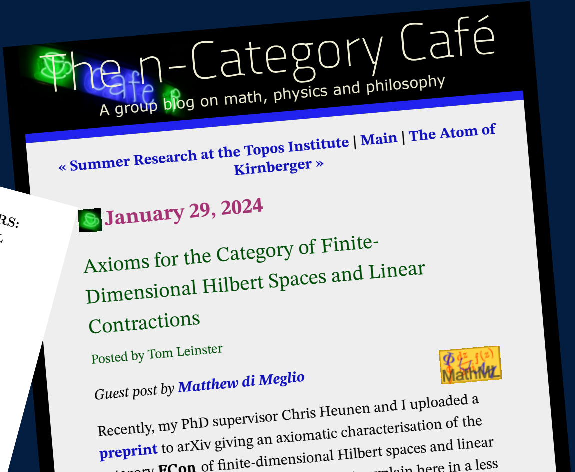
DAGGER CATEGORIES AND THE COMPLEX NUMBERS: AXIOMS FOR THE CATEGORY OF FINITE-DIMENSIONAL HILBERT SPACES AND LINEAR CONTRACTIONS

MATTHEW DI MEGLIO AND CHRIS HEUNEN

ABSTRACT. We characterise the category of finite-dimensional Hilbert spaces and linear contractions using simple category-theoretic axioms that do not refer to norms, continuity, dimension, or real numbers. Our proof directly relates limits in category theory to limits in analysis, using a new variant of the classical characterisation of the real numbers instead of Solèr's theorem.

1. INTRODUCTION

The category **Hilb** of Hilbert spaces and bounded linear maps and the category **Con** of Hilbert spaces and linear contractions were both recently char…



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January 29, 2024

Axioms for the Category of Finite-Dimensional Hilbert Spaces and Linear Contractions

Posted by Tom Leinster

Guest post by ***Matthew di Meglio***

Recently, my PhD supervisor Chris Heunen and I uploaded a **preprint** to arXiv giving an axiomatic characterisation of the category **FCon** of finite-dimensional Hilbert spaces and linear…

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Bonus

All axioms for FCon

1. O is initial,
2. $i_1 = \left(I \cong I \oplus O \xrightarrow{1 \oplus 0} I \oplus I\right)$ and $i_2 = \left(I \cong O \oplus I \xrightarrow{0 \oplus 1} I \oplus I\right)$ are jointly epic,
3. $i_1{}^{\dagger} d \neq 0 \neq i_2{}^{\dagger} d$ for some $d \colon I \to I \oplus I$,
4. I is dagger simple,
5. I is a monoidal separator,

All axioms for FCon

6. if $x\colon A \to X$ and $y\colon A \to Y$ are epic, then $x^\dagger x = y^\dagger y$ if and only if $y = fx$ for some isomorphism $f\colon X \to Y$,
7. every parallel pair has a dagger equaliser,
8. every dagger monomorphism is a kernel,
9. every bounded sequential diagram has a colimit,
10. every object is dagger finite.

Dagger finiteness

- An object X is called ***dagger finite*** when, for each $f\colon X \to X$,

$$f^\dagger f = 1 \qquad \Longleftrightarrow \qquad f f^\dagger = 1$$

- Finite-dimensional implies dagger finite by rank-nullity
- The right-shift map $R\colon \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ satisfies

$$R(x_1, x_2, \ldots) = (0, x_1, \ldots) \qquad \text{and} \qquad R^\dagger(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

  so $R^\dagger R = 1$ and $RR^\dagger \neq 1$; hence $\ell_2(\mathbb{N})$ is not dagger finite
- $\ell_2(\mathbb{N})$ embeds isometrically in all infinite-dimensional Hilbert spaces, so no infinite-dimensional Hilbert space is dagger finite