Enriched Symmetric Lenses and Enriched Bisimulations

(Extended abstract)

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We define the new notions of *enriched symmetric lens* and *enriched bisimulation* and show, under sufficient assumptions on the enrichment base, that both can be represented by spans of enriched asymmetric lenses. Symmetric delta lenses, bisimulations of Kripke frames, and strong and weak bisimulations of labelled transition systems may all be recovered by choosing an appropriate enrichment base. This work invites one to try to lift related ideas from modal logic and concurrency to the enriched setting, and contemplate the implications for other concrete instances such as Lawvere metric spaces.

Bisimulations are a relation-like kind of morphism between graph-like structures that encode a weaker notion of equivalence than isomorphisms. In computer science, bisimulations between labelled transition systems were introduced by Park [16], who is also credited by Milner [15] for the realisation that, in concurrency theory, their construction is an efficient method for proving observational equivalence of processes. They have since also become an important tool in several other areas of computer science, such as programming languages, databases and verification. In logic, bisimulations between Kripke models were introduced by van Benthem [2, 3, 21], who originally called them *p-relations* and later renamed them to *zigzag relations*; they are essential to his characterisation of the fragment of first-order logic that corresponds to modal logic, now known as the *van Benthem Characterisation Theorem*. Since then, bisimulations have been used extensively in the study of modal logic. We will use the adjective *functional* to describe bisimulations whose underlying relation is a function. In computer science, functional bisimulations between labelled transition systems have been called *abstraction homomorphisms* [5] and *pure morphisms* [1]. In logic, functional bisimulations between Kripke models are usually called *p-morphisms* [20, 18]—a shortening of the original name *pseudo-epimorphism* [19], but have also been called *zigzag morphisms* [21] and *bounded morphisms* [11]. In both disciplines, interest in functional bisimulations is due to the ability to represent bisimulations as spans of functional bisimulations.[1]

A *bidirectional transformation* is both a specification of the consistency of a pair of related systems and a protocol for restoring their consistency after updates [10]. *Symmetric delta lenses* are a kind of morphism between categories, introduced by Diskin et al. [9] to model bidirectional transformations in which consistency restoration takes as input specifically which update (delta) occurred rather than merely the state resulting from the update. In this context, the objects and morphisms of a category encode, respectively, the states and updates of a system. Just as bisimulations can be represented by spans of functional bisimulations, symmetric delta lenses can be represented by spans of *asymmetric delta lenses*. An asymmetric delta lens is another kind of morphism between categories, also introduced by Diskin et al. [8] to model bidirectional transformations, but in this case, specifically those where the state of one of the systems can be fully derived from that of the other.

In this talk, we will give a mathematical explanation for the similarities between symmetric delta lenses and several different kinds of bisimulations, using ideas from enriched category theory. We begin with the recently introduced notion of *enriched asymmetric lens* [7]—a generalisation of asymmetric delta lens to the setting of categories enriched in a distributive monoidal category that also makes sense for graphs enriched in a category with small coproducts. The connection to bisimulation comes from choosing the right base of enrichment.

[1] This paragraph draws from Sangiorgi's account of the history of bisimulation [17].

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Example 1. Functional bisimulations between Kripke frames are precisely asymmetric lenses between graphs enriched in the *interval category*, that is, the category with two objects $\top$ and $\bot$ and a morphism $\bot \to \top$.

Example 2. Functional strong bisimulations between transition systems labelled in a set L are precisely asymmetric lenses between graphs enriched in the *free suplattice* $\mathcal{P}(L)$ on L.

Example 3. We may view a transition system labelled in $A \sqcup \{\tau\}$, where A is a set of actions and τ is the distinguished *silent action*, as a category enriched in the *free quantale* $\mathcal{P}(A^*)$ on A as follows: starting with the corresponding graph enriched in $\mathcal{P}(A \sqcup \{\tau\})$, change base along the canonical suplattice morphism $\mathcal{P}(A \sqcup \{\tau\}) \to \mathcal{P}(A^*)$ that maps τ to the unit of the monoid A^*, and then take the free enriched category on the resulting enriched graph. Functional weak bisimulations between transition systems labelled in $A \sqcup \{\tau\}$ are precisely asymmetric lenses between the corresponding categories enriched in $\mathcal{P}(A^*)$.

We will define the new notions of *symmetric lens* and *bisimulation* for suitably enriched graphs and categories, and show, under certain assumptions, that these are in bijection with equivalence classes of spans of enriched asymmetric lenses. In order to be able to compose spans of enriched asymmetric lenses, we will also introduce an enriched analogue of *proxy pullback* [6, 13, 14]. If the enrichment base is *locally completely distributive*, or, equivalently [4], has all pullbacks and *universal* coproducts, then all cospans of enriched asymmetric lenses have an enriched proxy pullback.

Definition 4. A *symmetric lens* $R\colon \mathcal{A} \to \mathcal{B}$ of categories enriched in a distributive monoidal category consists of a span $\big(p_1, \mathrm{corr}(R), p_2\big)$ of functions from $\mathrm{obj}(\mathcal{A})$ to $\mathrm{obj}(\mathcal{B})$—the elements of $\mathrm{corr}(R)$ are called *correspondences*—and families of morphisms

$$\overleftarrow{R}_{r,b}\colon \mathcal{B}(p_2 r, b) \to \sum_{x \in p_2^{-1}\{b\}} \mathcal{A}(p_1 r, p_1 x) \qquad \text{and} \qquad \overrightarrow{R}_{r,a}\colon \mathcal{A}(p_1 r, a) \to \sum_{x \in p_1{}^{-1}\{a\}} \mathcal{B}(p_2 r, p_2 x),$$

called *leftward* and *rightward propagators*, that preserve identity elements and composition maps.

The analogous notion for graphs enriched in a category with small coproducts is defined similarly. Symmetric lenses between categories enriched in $\mathcal{S}\mathrm{et}$ are precisely symmetric delta lenses. We identify enriched symmetric lenses that have the same leftward and rightward actions on hom objects, and do the same for spans of asymmetric lenses; the equivalence relations are analogous to those defined by Johnson and Rosebrugh for symmetric delta lenses and spans of asymmetric delta lenses [12]. Under this identification, symmetric delta lenses and spans of asymmetric delta lenses both form categories.

Proposition 5. *Let $\mathcal{V}$ be a finitely complete distributive monoidal category in which coproducts are universal. The categories of symmetric lenses enriched in $\mathcal{V}$ and of spans of asymmetric lenses enriched in $\mathcal{V}$ are isomorphic.*

Definition 6. A *bisimulation* $R\colon \mathcal{A} \to \mathcal{B}$ of suplattice-enriched graphs is a relation $R \subseteq \mathrm{obj}(\mathcal{A}) \times \mathrm{obj}(\mathcal{B})$ such that if $(a,b) \in R$ then

$$\mathcal{B}(b', b) \leqslant \sup_{a \in R^{-1} b} \mathcal{A}(a', a) \qquad \text{and} \qquad \mathcal{A}(a', a) \leqslant \sup_{b \in Ra} \mathcal{B}(b', b).$$

A *bisimulation* of quantale-enriched categories is a bisimulation between the underlying enriched graphs.

As quantales are thin categories, quantale-enriched bisimulations already preserve identity elements and composition maps. For the particular enrichment bases mentioned earlier, we recover bisimulation of Kripke frames and strong and weak bisimulation of labelled transition systems.

Proposition 7. *Let $\mathcal{V}$ be a completely distributive quantale. The categories of bisimulations enriched in $\mathcal{V}$ and of spans of asymmetric lenses enriched in $\mathcal{V}$ are isomorphic.*

Proposition 7 is the enriched-category analogue of the earlier observation that bisimulations can be represented as spans of functional bisimulations. Combining Proposition 7 and Proposition 5, we see that, for suitable bases of enrichment, enriched bisimulations and enriched symmetric lenses coincide. This new-found link between enriched category theory and bisimulations elicits several interesting questions for future work. For example: What can be said about logics valued more generally in any quantale-enriched category? Do spans of enriched asymmetric lenses give rise to a sensible enriched notion of bisimilarity of objects? How does this work relate to the coalgebraic approach to bisimulation?

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