# BYPASSING SOLÈR'S THEOREM

## The key to axiomatising dagger categories of finite-dimensional Hilbert spaces

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Quantum mechanics is founded on the theory of Hilbert spaces and bounded linear maps.

The quantum reconstruction programme seeks more-intuitive mathematical foundations for quantum mechanics.

The category-theoretic approach identifies the essential structures and properties of categories of Hilbert spaces.

These should eventually inform the design of programming languages for quantum computers.

## THEOREM (Heunen and Kornell):

A monoidal dagger category in which

- finite dagger biproducts exist
- dagger equalisers exist
- monoidal unit is simple
- wide subcategory of dagger monos has directed colimits

⋮

is equivalent to **Hilb**.

identity-on-objects anti-involution $(-)^\dagger$ encodes Hermitian adjoints

enrichment in commutative monoids

$$\begin{tikzcd} X \arrow[r, "f+g"] \arrow[d, "\Delta"'] & Y \\ X \oplus X \arrow[r, "f \oplus g"'] & Y \oplus Y \arrow[u, "\nabla"'] \end{tikzcd}$$

$\mathbb{I} := \mathrm{Hom}(I, I)$ is a field and $\mathrm{char}(\mathbb{I}) = 0$

$\mathbb{I}$ is Dedekind complete

# Solèr's Theorem:

Let $X$ be an orthomodular space over an involutive division ring $\mathbb{K}$. If $X$ has an infinite orthonormal subset, then $\mathbb{K} \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $X$ is a Hilbert space.

$$\begin{tikzcd} I \arrow[r, "i_1"] \arrow[drrrrr, bend right] & I^{\oplus 2} \arrow[r, "i_{1,2}"] \arrow[drrrr, bend right] & I^{\oplus 3} \arrow[r, "i_{1,2,3}"] \arrow[drrr, bend right] & I^{\oplus 4} \arrow[r] \arrow[drr] & \cdots \\ & & & & & \ell_2(I) \end{tikzcd}$$

This idea will not work for <u>FDHilb</u>.

**GOAL:**

Prove that $\mathbb{I}$ is $\mathbb{R}$ or $\mathbb{C}$ without Solèr's theorem.

Non-negatives contain $1$ and closed under $+, \cdot, (-)^{-1}$
Every element is a difference of non-negatives

**Theorem** (De Marr 1967):

All $\underbrace{\text{Dedekind } \sigma\text{-complete}}$ $\overbrace{\text{partially ordered fields}}$ are isomorphic to $\mathbb{R}$.

Non-negatives have infima of
non-increasing sequences

## Proposition:

$\mathbb{I}_{SA} := \{z \in \mathbb{I} : z = z^\dagger\}$ is a partially ordered field with $a \leqslant b \Leftrightarrow b - a = x^\dagger x$ for some $x : I \to X$.

## Proof:

$a \in \mathbb{I}_{SA}$

$x : I \to X$

$y : I \to Y$

$a^2 = a^\dagger a$

$a = \frac{1}{4}(a+2)^2 - \frac{1}{4}(a^2+4)$

$1 = 1^\dagger 1$

$x^\dagger x \cdot y^\dagger y = (x \otimes y)^\dagger (x \otimes y)$

$x^\dagger x + y^\dagger y = \langle x, y\rangle^\dagger \langle x, y \rangle$

$(x^\dagger x)^{-1} = (x^\dagger x)^{-2} x^\dagger x$

# Proposition:

$\mathbb{I}_{SA}$ is Dedekind $\sigma$-complete if wide subcategory of $\underbrace{\text{contractions}}_{f\colon X \to Y \text{ such that } f^\dagger f + \bar{f}^\dagger \bar{f} = 1 \text{ for some } \bar{f}\colon X \to \bar{Y}}$ has directed colimits.

**Lemma:** If $a > 0$, then $a = x^\dagger x$ for some isomorphism $x\colon I \to X$.

**Proof:** $a = y^\dagger y$ for some $y\colon I \to Y$

$$\begin{tikzcd}
I \arrow[rr, "y"] \arrow[dr, dashed, "x"', "\cong"] & & Y \arrow[r, shift left, "y(y^\dagger y)^{-1} y^\dagger"] \arrow[r, shift right, "1"'] & Y \\
& X \arrow[ur, "\underbrace{e}_{\text{dagger equaliser}}"'] & &
\end{tikzcd}$$

$$\begin{aligned} x^\dagger x &= x^\dagger e^\dagger e x \\ &= y^\dagger y \\ &= a \end{aligned}$$

# Proof:

$$x_1^\dagger x_1 \geqslant x_2^\dagger x_2 \geqslant \cdots$$

$$x_j : I \to X_j \text{ iso}$$

$$1 = x_j^{-\dagger} x_j^\dagger x_j x_j^{-1} \geqslant x_j^{-\dagger} x_{j+1}^\dagger \underbrace{x_{j+1} x_j^{-1}}_{\text{contraction}}$$

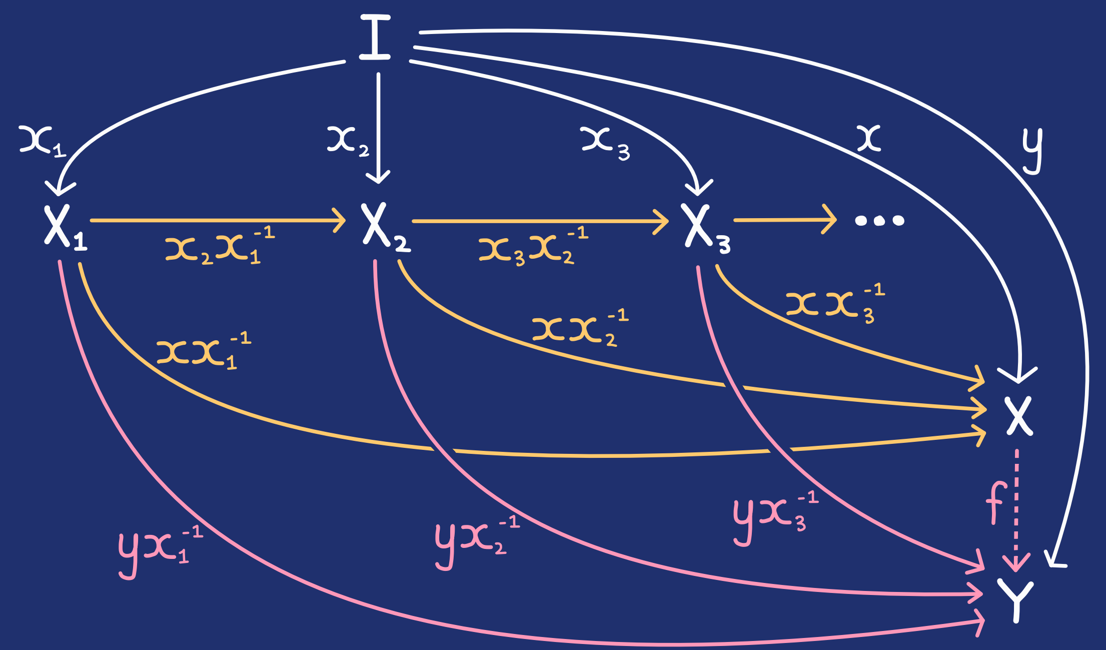
$$\underbrace{x^\dagger x}_{\text{lower bound}} = x_j^\dagger (x x_j^{-1})^\dagger (x x_j^{-1}) x_j \leqslant x_j^\dagger x_j$$

$$y : I \to Y$$

$$x_j^\dagger x_j \geqslant y^\dagger y$$

$$1 = x_j^{-\dagger} x_j^\dagger x_j x_j^{-1} \geqslant x_j^{-\dagger} y^\dagger \underbrace{y x_j^{-1}}_{\text{contraction}}$$

$$\begin{aligned}\underbrace{x^\dagger x}_{\text{greatest lower bound}} &\geqslant x^\dagger f^\dagger f x \\ &= x_1^\dagger (x x_1^{-1})^\dagger f^\dagger f (x x_1^{-1}) x_1 \\ &= x_1^\dagger (y x_1^{-1})^\dagger (y x_1^{-1}) x_1 \\ &= y^\dagger y\end{aligned}$$



## Theorem:

$\mathbb{I} \simeq \mathbb{R}$ or $\mathbb{I} \simeq \mathbb{C}$ if wide subcategory of contractions has directed colimits.

## Proof:

$\mathbb{I}_{SA} \simeq \mathbb{R}$ by De Marr's theorem. Suppose that $u \in \mathbb{I} \setminus \mathbb{I}_{SA}$. Let

$$i = \frac{u - u^{\dagger}}{\sqrt{-(u-u^{\dagger})^2}}.$$

Then $\{1, i\}$ is a basis for $\mathbb{I}$ over $\mathbb{I}_{SA}$ and $i^2 = -1$.

## Question:

Can directed colimits of contractions be constructed from those of dagger monos?