Bypassing Solèr's Theorem: The Key to Axiomatising Dagger Categories of Finite-Dimensional Hilbert Spaces

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Quantum computers, by exploiting the strangeness of quantum mechanics, are capable of solving certain problems using significantly less time or memory than ordinary computers. On the other hand, it is difficult to know for which problems quantum computers will be helpful, and even more difficult to design and implement quantum algorithms that actually achieve an improvement in run time or memory usage. We would greatly benefit from a new generation of quantum programming languages with better abstractions to facilitate the conception of quantum algorithms that fully harness the power of quantum computers.

One line of research that should inform the design of this new generation of quantum programming languages is the *quantum reconstruction programme* [13, 4, 8], that is, the long-running search for more-intuitive alternatives to Hilbert spaces for the mathematical foundations of quantum mechanics. The main alternatives that have emerged so far are the theory of operator algebras [10], the theory of orthomodular lattices [9, 11], and, more recently, category theory [1, 14, 3, 5, 16, 6]. The goal of the category-theoretic approach is to find nice structures and axioms to impose on a category to force its equivalence with some category of Hilbert spaces that is relevant to quantum computing, such as the category of finite-dimensional Hilbert spaces and unitary maps.

Vicary almost achieved this goal for the category of Hilbert spaces and bounded linear maps with his article on the completeness of dagger categories and the complex numbers [16]. A *dagger category* is a category equipped with an involutive contravariant endofunctor called the dagger—an abstraction of the the operation that sends a bounded linear map between Hilbert spaces to its Hermitian adjoint. Unfortunately, Vicary's explanation of why the semiring of scalars of a monoidal dagger category satisfying his axioms should be the field of real or complex numbers is unsatisfactory as it assumes that the self-adjoint scalars are Dedekind complete rather than deducing this fact from axioms of a more category-theoretic nature.

Heunen and Kornell, building on the work of Vicary and others, finally succeeded in Vicary's endeavour, giving a list of simple category-theoretic axioms that completely characterise the monoidal dagger category of Hilbert spaces and bounded linear maps [6]. Their key idea was to appeal to Solèr's theorem [15, 12] to deduce that the scalar field is either the real or complex numbers. This theorem is about *orthomodular spaces*—a not so well-known generalisation of Hilbert spaces—that have an infinite orthonormal subset. They construct such a space using the new assumption that the wide subcategory of dagger monomorphisms has directed colimits. Building on this result, Heunen, Kornell and van der Schaaf also gave similar axioms for the category of Hilbert spaces and linear contractions [7], taking us one step closer to the category of Hilbert spaces and unitary maps.

The desire to axiomatise categories of finite-dimensional Hilbert spaces stems from the fact that quantum computers can have only finitely many qubits and so their state spaces are finite dimensional. The main obstruction to adapting existing ideas to give, for example, axioms for the category of finite-dimensional Hilbert spaces and linear maps, or the category of finite-dimensional Hilbert spaces and linear contractions, is the dependence on Solèr's theorem to deduce that the scalars are the real or complex numbers. This is because Solèr's theorem is inherently about infinite-dimensional spaces. Whilst it may be possible to reduce axiomatising a category of finite-dimensional Hilbert spaces to axiomatising the corresponding category of all Hilbert spaces by some kind of category completion in a similar way to how Heunen, Kornell and van der Schaaf reduced

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axiomatising Hilbert spaces and linear contractions to axiomatising Hilbert spaces and all bounded linear maps by localising the scalars, the authors' attempts to do this have so far proved fruitless.

In this talk, I will explain an alternative approach that entirely bypasses Solèr's theorem. In particular, I will prove that the scalar field is indeed either the field of real or complex numbers by constructing infima of decreasing sequences of positive scalars explicitly in terms of certain directed colimits of contractions. The directed colimits of contractions that we need turn out to exist even in the categories of finite-dimensional Hilbert spaces. A forthcoming article by the authors of this abstract will use this proof to give axioms for the category of finite-dimensional Hilbert spaces and linear maps. Another forthcoming article by these authors and Kornell will use similar ideas to give axioms for the category of finite-dimensional Hilbert spaces and linear contractions. This talk will be a shortened version of the talk given at the Edinburgh Category Theory seminar [2]; see the linked slides for technical details.

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