Coequalisers under the lens

Matthew Di Meglio

MACQUARIE University

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What is a lens?

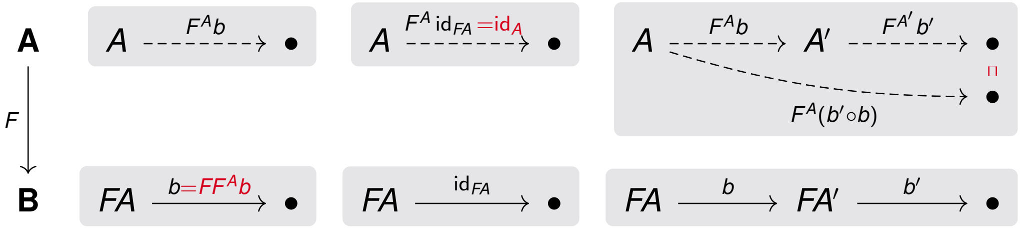
	Example	Model
system	database, view	category
state	records in each table	object
transition	insert record, update record, delete record	morphism
synchronisation protocol	solution to view-update problem	(delta) lens

What is a lens?

A ***lens*** $F\colon \mathbf{A} \to \mathbf{B}$ consists of

- a ***get functor*** $F\colon \mathbf{A} \to \mathbf{B}$, and
- for all A in $\mathbf{A}$ and $b\colon FA \to \bullet$ in $\mathbf{B}$, a ***lift*** $F^A b\colon A \to \bullet$ in $\mathbf{A}$ of b to A,

such that



(PutGet) (PutId) (PutPut)

The category of lenses

- Small categories and lenses form a category **Lens**
- Chollet et al. initiated a study of the categorical properties of **Lens**
- No reason to expect **Lens** would have nice properties but it does
- Functor $U\colon \mathbf{Lens} \to \mathbf{Cat}$ sending a lens to its get functor helpful
- Proved Chollet et al.'s conjectures about monos and epis
- Characterisation of epis enabled a start on studying coequalisers

Epis in **Lens** are nicer than epis in **Cat**

e is ***epic*** if it is right cancellable ($h_1 e = h_2 e$ implies $h_1 = h_2$)

Remark

In **Cat**

epic $\implies$ surjective on objects

epic $\not\Longrightarrow$ surjective on morphisms

$$\text{epic} \impliedby \begin{cases} \text{surjective on objects} \\ + \\ \text{surjective on morphisms} \end{cases}$$

Proposition

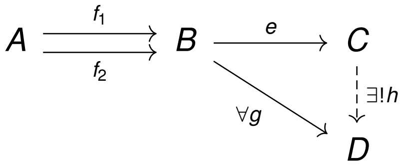
In **Lens**

epic $\iff$ *surjective on objects*

$\iff$ *surjective on morphisms*

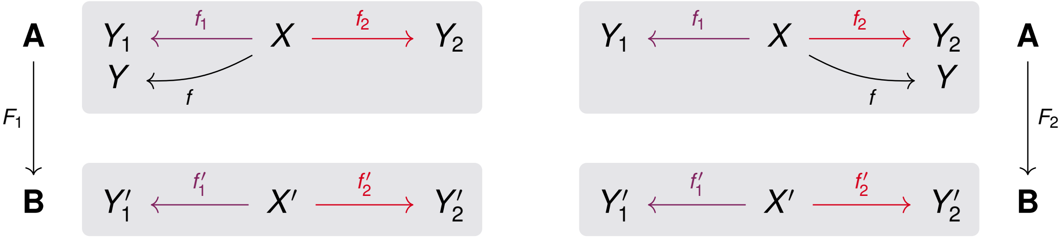
Coequalisers

e ***coequalises*** f_1 and f_2 if it is their universal ***cofork***



- **Cat** has all coequalisers, but they aren't usually nice to describe
- **Lens** doesn't have all coequalisers, but some are nicer to describe
- Coequalisers are always epic

Not all coequalisers in **Lens** exist



Coequalisers in **Lens** above coequalisers in **Cat**

Lemma

The get functor of every epic lens coequalises its kernel pair in **Cat**.

Theorem

Every epic lens coequalises its imported kernel pair in **Lens**.

Corollary

The lenses left orthogonal to all monic lenses are the epic lenses.

Lemma

A lens is monic if and only if it is injective on objects.

Theorem

U creates pushouts of monic lenses with discrete opfibrations.

Corollary

Every monic lens equalises its cokernel pair in **Lens**.

Conclusion

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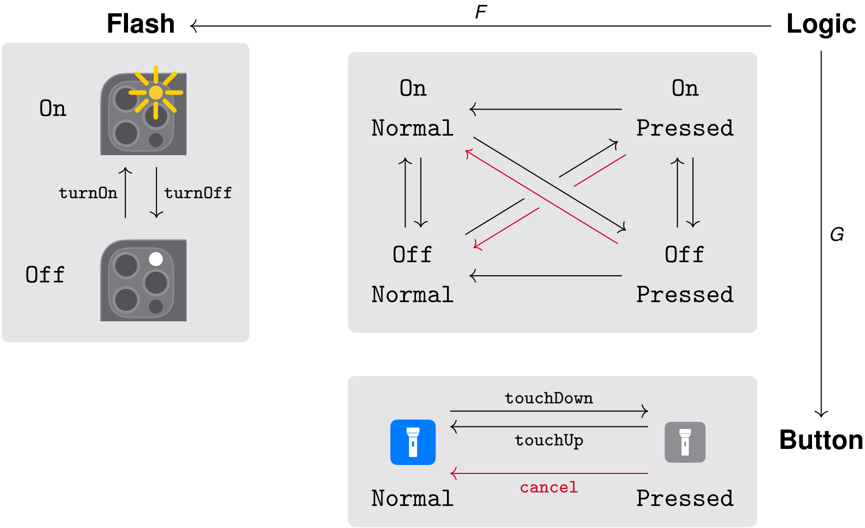
Summary

- Epis in **Lens** are nicer than those in **Cat**
- Epic lens characterisation enabled start studying coequalisers in **Lens**
- **Lens** doesn't have all coequalisers, nor does U reflect/preserve them
- There are classes of coequalisers which are preserved/created by U

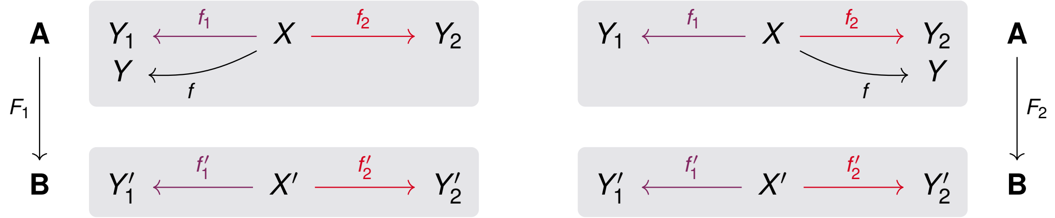
Future work

- Completely characterise pullbacks and coequalisers in **Lens**
- Study category of symmetric lenses via properties of **Lens**
- General theory of categories of morphisms with extra structure?

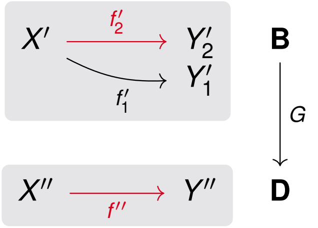
Toy example


Flash
F
Logic
On
turnOn
turnOff
Off
On
Normal
On
Pressed
Off
Normal
Off
Pressed
G
touchDown
touchUp
cancel
Normal
Pressed
Button

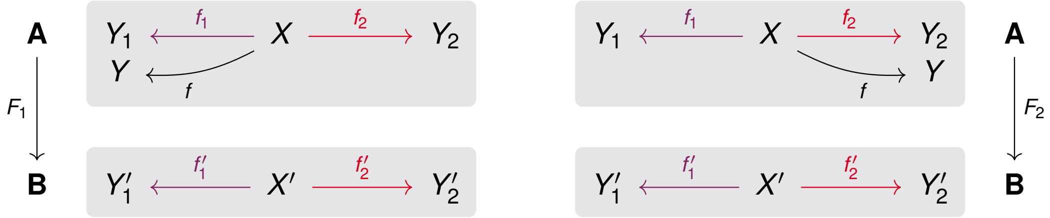
Lens doesn't have all coequalisers

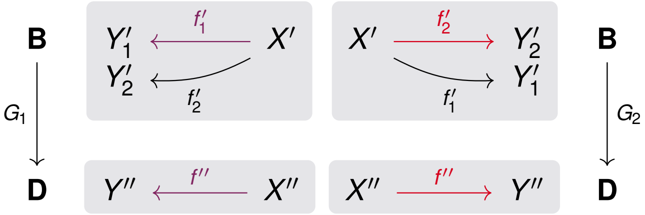


- Assume F_1 and F_2 have a coequaliser $E \colon \mathbf{B} \to \mathbf{C}$
- $Ef'_1 = Ef'_2$ as E coforks F_1 and F_2
- Let $H \colon \mathbf{C} \to \mathbf{D}$ be the unique lens with $G = H \circ E$
- $GX' \neq GY'_1$ implies $EX' \neq EY'_1$
- EX', EY'_1 and Ef'_1 is all of $\mathbf{C}$ as E is epic
- Thus H is iso, so G also coequalises F_1 and F_2



U doesn't reflect coequalisers





- G_1 and G_2 are the only lenses that U maps to the coequaliser in **Cat** of UF_1 and UF_2
- No lens $H\colon \mathbf{D} \to \mathbf{D}$ such that $G_1 = H \circ G_2$