The category of asymmetric lenses and its proxy pullbacks

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Abstract. We study the categorical properties of the category of small categories and asymmetric delta lenses, continuing the work begun by Chollet et al. at the Applied Category Theory Adjoint School 2020. We give complete elementary characterisations of the monic and epic lenses, confirming several of Chollet et al.'s conjectures. We also initiate the study of lens coequalisers, ultimately showing that every epic lens is regular, and that discrete opfibrations have pushouts along monic lenses.

An important construction for proving many of these results is Johnson and Rosebrugh's *"pullback"* of lenses, which we call the *proxy pullback* of lenses. We give a new treatment of the proxy pullback in terms of *compatibility*—a stronger notion of commutativity for squares of lenses. We also prove that the proxy pullback has several pullback-like properties, including an analogue of the well-known pullback pasting lemma. The proxy pullback is sometimes, but not always, a real pullback. Using new notions of *sync-minimal* and *independent* lens spans, we characterise when a lens span that forms a commuting square with a lens cospan has a comparison lens to a proxy pullback of the cospan.

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Contents

CHAPTER 1

Introduction

The term *lens* has become widely used for naming mathematical structures that capture fundamental features of bidirectional transformations. A *bidirectional transformation* is a specification of when the joint state of two systems should be regarded as consistent, together with a protocol for updating each system to restore consistency in response to a change in the other [16]. The study of bidirectional transformations goes back as far as 1981 with Bancilhon and Spyratos' work on the view-update problem for databases [3]. The view-update problem is about *asymmetric* bidirectional transformations; those where the state of one of the systems, called the *view*, is completely determined by that of the other, called the *source*. Bidirectional transformations also arise in many other contexts across computer science, such as when programming with complex data structures and when linking user interfaces to data models.

An *asymmetric state-based lens* is a mathematical encoding of an asymmetric bidirectional transformation in which the consistency restoration updates to the source are assumed to be dependent only on the old source state and the updated view state. If S is the set of source states and V is the set of view states, such a lens consists of a *get function* $S \to V$ and a *put function* $S \times V \to S$ that are often required to satisfy certain axioms. The earliest known account of asymmetric state-based lenses may be found in Oles' PhD thesis [25, Chapter VI], where they are called *extensions* of *store shapes*; they are a key ingredient in Oles' semantics for an imperative stack-based programming language with block-scoped variables because they capture the essential properties of a data store that changes shape as variables come into and go out of scope. All recent notions of lens, as well as the name, may be traced back to the work of Pierce et al. [14]; they proposed variants of asymmetric state based lenses for modelling bidirectional transformations on tree-structured data, and they also introduced—with their lens combinators—the idea of domain specific languages for building lenses compositionally.

Diskin et al. highlighted the inadequacy of state-based lenses as a general mathematical model for bidirectional transformations [12], providing several examples of situations in which consistency restoration would benefit from knowing more about each change to the view than just the view's new state. In an *asymmetric delta lens*, their proposed alternative, systems are modelled as categories of states and transitions (deltas) rather than simply as sets of states. The put operation takes as input specifically which transition occurred in the view rather than just the end state of that transition, and it likewise gives as output a transition in the source rather than just the end state of that transition. In this thesis we will use the terms *lens* and *asymmetric lens* as synonyms for the term *asymmetric delta lens*.

Application of category theory to the study of lenses has already proved fruitful. Johnson and Rosebrugh's research program [17, 19, 20] has enabled a unified treatment of symmetric and asymmetric lenses, with the perspective that a symmetric lens is an equivalence class of spans of asymmetric lenses. Ahman and Uustalu's observation that asymmetric lenses are compatible functor cofunctor pairs [2], and Clarke's generalisation of lenses to the internal category theory setting [10], have enabled an abstract diagrammatic approach to these lenses [9], in which we may profit from the already well-developed theory of functors and discrete opfibrations.

Chollet et al. [8], a group from the Applied Category Theory Adjoint School 2020, initiated the effort to better understand the category $\mathcal{L}ens$ of small categories and lenses by

- showing that this category has equalisers and coproducts,
- giving sufficient conditions for a lens to be monic and for a lens to be epic,
- giving sufficient conditions for the proxy pullback of lenses (they call it the imported pullback) to be a real pullback, and
- showing that $\mathcal{L}ens$ is actually extensive.

Clarke presented their progress to the Australian Category Seminar in February 2021, along with several conjectures, including that their sufficient conditions for a lens to be monic and also those for a lens to be epic are actually also necessary conditions.

The starting point for this thesis was the author's realisation that Chollet et al.'s mono conjecture does indeed hold, and actually has a simple proof using the proxy pullback. This also inspired a proof for their epi conjecture, which in turn enabled progress in studying the coequalisers in $\mathcal{L}ens$. Despite $\mathcal{L}ens$ not having all coequalisers, nor the forgetful functor from $\mathcal{L}ens$ to $\mathcal{C}at$ preserving or reflecting them, the author obtained two positive results, namely that

- pushouts of discrete opfibrations along monic lenses exist, and
- every epic lens is proxy effective, that is, coequalises its proxy kernel pair.

The author presented his thesis work up to this point, including the results about monos and epis and those about coequalisers, at the Applied Category Theory 2021 conference. An associated article [11] will appear in the conference proceedings.

The latter of the two positive results above is especially surprising since there are epis in $\mathcal{C}at$ that are not effective epis. It also suggests that proxy pullbacks may actually be a good substitute for real pullbacks in more situations than one might expect. Indeed, checking whether $\mathcal{L}ens$ has proxy analogues of common pullback-related properties revealed several interesting results. One such result is a proxy analogue of the well-known *pullback pasting lemma*. This is quite surprising because the pullback pasting lemma is usually proved using the universal property of the pullback, one that proxy pullbacks do not in general possess. Another one is that $\mathcal{L}ens$ has a proxy-pullback-stable regular-epi mono orthogonal factorisation system—we might say that $\mathcal{L}ens$ is a *proxy-regular category*. This is again surprising as $\mathcal{C}at$ is not a regular category.

Outline

In Chapter 2, as well as establishing notation used throughout this thesis, we define the notions of cofunctor and lens, and their respective categories $\mathcal{C}of$ and $\mathcal{L}ens$. We also recall several properties of discrete opfibrations and split opfibrations—these are special kinds of lenses.

In Chapter 3, we give a new treatment of proxy pullbacks that is more amenable to a category theoretic style of argument. This treatment begins with a new definition of the proxy pullback in terms of *compatibility*—a notion of commutativity for squares of lenses that is stronger than the usual notion of commutativity. With our new definition, a given lens cospan could have many proxy pullbacks, but there is always a unique span isomorphism between any two of them. Additionally, the forgetful functor from $\mathcal{L}ens$ to $\mathcal{C}at$ in some sense creates proxy pullbacks in $\mathcal{L}ens$ from real pullbacks in $\mathcal{C}at$; in particular, every lens cospan has a proxy pullback. We also prove several other pullback-like properties of the proxy pullback, including

- that a *compatible-lens transformation* (a kind of natural transformation) between two lens cospans extends uniquely to one between proxy-pullback squares on the cospans; and,
- the aforementioned proxy analogue of the pullback pasting lemma.

In Chapter 4, inspired by the approaches of Böhm [6] and Simpson [29] to pullback-like constructions in other categories, we characterise when a lens span that forms a commuting square with a lens cospan has a comparison lens to a proxy pullback of the cospan. With the new notions of *sync-minimal* and *independent* lens spans, we prove that

- the existence of such a comparison lens necessitates that the lens span be independent and be compatible with the cospan, and
- a proxy pullback of a lens cospan is terminal amongst the independent spans that are compatible with the cospan if and only if the proxy-pullback span is sync-minimal.

In particular, a proxy pullback of a lens cospan is a real pullback if and only if the proxy-pullback span is sync minimal and all lens spans that form commuting squares with the cospan are independent and are compatible with the cospan. Specialising these results for proxy products, we also show that a proxy product of two categories is a real product if and only if at least one of the two categories is a discrete category, confirming another of Chollet et al.'s conjectures.

In Chapter 5, we prove the conjecture by Chollet et al. [8] that the forgetful functor from the category of lenses to the category of functors preserves monos. Together with their result that it reflects monos, we deduce that the monic lenses are the unique lenses on cosieves; these are equivalently the out-degree-zero subcategory inclusion functors. We also provide a proof, simpler than the original one sketched by Lack, that the forgetful functor preserves epis.

In Chapter 6, we initiate the study of coequalisers of lenses. We begin with examples of how they are not as well behaved as one might hope; specifically, not all parallel pairs of lenses have coequalisers, and the forgetful functor from $\mathcal{L}ens$ to $\mathcal{C}at$ neither preserves nor reflects all coequalisers. We then prove the main result of this chapter, Theorem 6.6, which is about the coequalisers that are actually reflected by the forgetful functor. We use Theorem 6.6 to show that the category of lenses has pushouts of discrete opfibrations along monos, and also that every epic lens is regular. The former enables us to also show that every monic lens is effective.

An early version of Chapters 5 and 6 will appear in the Applied Category Theory 2021 conference proceedings in the article *Coequalisers under the lens* [11]. Although the work on proxy pullbacks in Chapters 3 and 4 is more recent, it seems most natural to present it first.

CHAPTER 2

Background

2.1. Notation

Application of functions (functors, etc.) is written by juxtaposing the function name with its argument, and parentheses are only used when needed. Binary operators like $\circ$ have lower precedence than application, so an expression like $Fa \circ Fb$ parses as $(Fa) \circ (Fb)$.

Let $\mathcal{C}at$ denote the category whose objects are small categories and whose morphisms are functors. Categories with boldface names $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc. are always small. We write $|\mathbf{C}|$ for the set of objects of a small category $\mathbf{C}$, and, for all $X, Y \in |\mathbf{C}|$, we write $\mathbf{C}(X, Y)$ for the set of morphisms of $\mathbf{C}$ from X to Y. For each $X \in |\mathbf{C}|$, we write $\mathbf{C}(X, *)$ for the set $\bigsqcup_{Y \in |\mathbf{C}|} \mathbf{C}(X, Y)$ of all morphisms in $\mathbf{C}$ out of X. We write $\operatorname{src} f$ and $\operatorname{tgt} f$ for, respectively, the source and target of a morphism f. We also write $f\colon X \to Y$ to say that $X, Y \in |\mathbf{C}|$ and $f \in \mathbf{C}(X, Y)$. The composite of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted $g \circ f$.

The category with a single object 0 and no non-identity morphisms, also known as the *terminal category*, is denoted $\mathbf{1}$. The category with two objects 0 and 1 and a single non-identity morphism, namely $u\colon 0 \to 1$, also known as the *interval category*, is denoted $\mathbf{2}$. The category with two objects 0 and 1 and two non-identity morphisms, namely $v\colon 0 \to 1$ and $v^{-1}\colon 1 \to 0$, also known as the *free living isomorphism*, is denoted $\mathbf{I}$. We will identify objects and morphisms of a small category $\mathbf{C}$ with the corresponding functors $\mathbf{1} \to \mathbf{C}$ and $\mathbf{2} \to \mathbf{C}$ respectively.

If the square

$$\begin{tikzcd} \mathbf{D} \arrow[r, "T"] \arrow[d, "S"'] & \mathbf{B} \arrow[d, "G"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{C} \end{tikzcd} \tag{1}$$

in $\mathcal{C}at$ is a pushout square and $F'\colon \mathbf{A} \to \mathbf{C}'$ and $G'\colon \mathbf{B} \to \mathbf{C}'$ are functors for which $F' \circ S = G' \circ T$, then we write $[F', G']$ for the functor $\mathbf{C} \to \mathbf{C}'$ induced from F' and G' by the universal property of the pushout. Similarly, if the square (1) in $\mathcal{C}at$ is a pullback square and $S'\colon \mathbf{D}' \to \mathbf{A}$ and $T'\colon \mathbf{D}' \to \mathbf{B}$ are functors for which $F \circ S' = G \circ T'$, then we write $\langle S', T' \rangle$ for the functor $\mathbf{D}' \to \mathbf{D}$ induced from S' and T' by the universal property of the pullback. By our above identification of objects with functors from $\mathbf{1}$, if $A \in |\mathbf{A}|$ and $B \in |\mathbf{B}|$ are such that $FA = GB$, then $\langle A, B \rangle$ is the object of $\mathbf{D}$ selected by the functor $\mathbf{1} \to \mathbf{D}$ induced by the universal property of the pullback from the functors $\mathbf{1} \to \mathbf{A}$ and $\mathbf{1} \to \mathbf{B}$ that respectively select the objects A and B.

2.2. Cofunctors and lenses

The definition of (asymmetric delta) lens most useful to us will be as a suitable pairing of a functor and a cofunctor [2]. Let us first recall the definition of a cofunctor [1, 10], specialised from the internal category theory setting to categories internal to $\mathcal{S}et$, i.e. small categories.

Definition 2.1. For small categories $\mathbf{A}$ and $\mathbf{B}$, a *cofunctor* $F\colon \mathbf{A} \to \mathbf{B}$ consists of

- a function $F\colon |\mathbf{A}| \to |\mathbf{B}|$, called the *object function*, and
- functions $F^A\colon \mathbf{B}(FA, *) \to \mathbf{A}(A, *)$ for all $A \in |\mathbf{A}|$, called *lifting functions*,

such that the equations

$$\underset{\text{(PutTgt)}}{F\,\mathrm{tgt}\,F^A b = \mathrm{tgt}\,b} \qquad \underset{\text{(PutId)}}{F^A\,\mathrm{id}_{FA} = \mathrm{id}_A} \qquad \underset{\text{(PutPut)}}{F^A(b' \circ b) = F^{A'}b' \circ F^A b}$$

hold whenever they are defined.

Warning 2.2. The notions of cofunctor and contravariant functor are distinct and unrelated. It is unfortunate that the name cofunctor is now entrenched in the literature as the similarity between the names cofunctor and contravariant functor is a common source of confusion. In Chapter 7 the author proposes that we rename cofunctor to *retrofunctor* in order to eliminate this point of confusion, although it is likely too late for such a name change to be widely adopted.

There is a category $\mathcal{C}of$ whose objects are small categories and whose morphisms are cofunctors. The composite $G \circ F$ of cofunctors $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ has as its object function the composite of the object functions of F and G, and has $(G \circ F)^A c = F^A G^{FA} c$ for all $A \in |\mathbf{A}|$ and all $c \in \mathbf{C}(GFA, *)$.

In the following definition of a lens, although we use the name of the lens to refer both to its get functor and its put cofunctor, the compatibility condition ensures that there is no ambiguity.

Definition 2.3. For small categories $\mathbf{A}$ and $\mathbf{B}$, a *lens* $F\colon \mathbf{A} \to \mathbf{B}$ consists of

- a functor $F\colon \mathbf{A} \to \mathbf{B}$, called the *get functor*, and
- a cofunctor $F\colon \mathbf{A} \to \mathbf{B}$, called the *put cofunctor*,

with same object functions, such that the equation

$$FF^A b = b \tag{PutGet}$$

holds whenever it is defined.

There is a category $\mathcal{L}ens$ whose objects are small categories and whose morphisms are lenses. Identity morphisms and composites in $\mathcal{L}ens$ come from those in $\mathcal{C}at$ and $\mathcal{C}of$ in the obvious way. There are also identity-on-objects functors

$$\mathcal{G}\colon \mathcal{L}ens \to \mathcal{C}at \qquad \text{and} \qquad \mathcal{P}\colon \mathcal{L}ens \to \mathcal{C}of$$

that respectively send a lens to its get functor and put cofunctor.

It will at times be convenient to draw functors, cofunctors and lenses all in a single diagram; we shall call such a diagram a *mixed diagram*. In a mixed diagram, the arrow always points in the direction of the action on objects whilst the solid arrow head decorations indicate the directions of the actions on morphisms. Thus the arrows

$$\mathbf{A} \longrightarrow \mathbf{B} \qquad\qquad \mathbf{A} \longrightarrow \mathbf{B} \qquad\qquad \mathbf{A} \longrightarrow \mathbf{B}$$

in a mixed diagram respectively represent a functor, a cofunctor and a lens. When we say that a mixed diagram *commutes*, we mean that

- the diagram of the object functions of all of the functors, cofunctors and lenses commutes;
- the diagram of all of the functors, including the get functors of the lenses, commutes; and
- the diagram of all of the cofunctors, including the put cofunctors of the lenses, commutes.

2.3. Discrete opfibrations and split opfibrations

Split opfibrations and discrete opfibrations are both important classes of lenses.

Definition 2.4. A functor $F\colon \mathbf{A} \to \mathbf{B}$ is a *discrete opfibration* if, for each $A \in |\mathbf{A}|$ and each $b \in \mathbf{B}(FA, *)$, there is a unique $a \in \mathbf{A}(A, *)$ such that $Fa = b$.

Definition 2.5. A lens $F\colon \mathbf{A} \to \mathbf{B}$ is a *discrete opfibration* if the equation

$$F^A F a = a \tag{GetPut}$$

holds for each $A \in |\mathbf{A}|$ and each $a \in \mathbf{A}(A, *)$.

Remark 2.6. The name GetPut has, in the past, been used for what is now called PutId. The author believes that our repurposing of the name GetPut is appropriate as the above equation is, in some sense, dual to the one for PutGet. The reader should note that lenses in general need not satisfy GetPut the way that we have defined it.

If $F\colon \mathbf{A} \to \mathbf{B}$ is a discrete opfibration, then there is a unique lens mapped by $\mathcal{G}$ to F, which we sometimes also refer to as F. A lens is a discrete opfibration if and only if its get functor is a discrete opfibration. Together, these results mean that we need not specify whether a discrete opfibration $F\colon \mathbf{A} \to \mathbf{B}$ is a functor or a lens, and we can use the name F in both functor and lens contexts without ambiguity.

Definition 2.7. For a functor $F\colon \mathbf{A} \to \mathbf{B}$, a morphism $f\colon X \to Y$ in $\mathbf{A}$ is F*-opcartesian* if, for all morphisms $f'\colon X \to Y'$ in $\mathbf{A}$ and all morphisms $v\colon FY \to FY'$ in $\mathbf{B}$ such that $Ff' = v \circ Ff$, there is a unique morphism $u\colon Y \to Y'$ in $\mathbf{A}$ such that $f' = u \circ f$ and $v = Fu$. For f to be *weakly* F*-opcartesian*, the property described in the previous sentence need only hold for $v = \mathrm{id}_{FY}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\forall f'"'] & Y \arrow[d, dashed, "\exists! u"] & & FX \arrow[r, "Ff"] \arrow[dr, "Ff'"'] & FY \arrow[d, "\forall v"] \\
& Y' \arrow[r, phantom, ""] & {} \arrow[r, mapsto, "F"'] & {} & FY'
\end{tikzcd}$$

$\mathbf{A}$ $\qquad\qquad$ $\mathbf{B}$

Definition 2.8. A lens $F\colon \mathbf{A} \to \mathbf{B}$ is a *split opfibration* if each morphism $F^A b$ is $\mathcal{G}F$-opcartesian.

It is well known that, for a functor, having opcartesian lifts is equivalent to having weakly opcartesian lifts that are closed under composition. In the following proposition, by starting with a *lens*, we have assumed, with the PutPut axiom, that the chosen lifts are closed under composition. Hence it suffices to ask that these chosen lifts be weakly opcartesian.

Proposition 2.9. *A lens $F\colon \mathbf{A} \to \mathbf{B}$ is a split opfibration if and only if, for all $a\colon A \to A'$ in $\mathbf{A}$, there is a unique $u\colon \operatorname{tgt} F^A Fa \to A'$ in $\mathbf{A}$ such that $a = u \circ F^A Fa$ and $Fu = \mathrm{id}_{FA'}$.*

$$\underset{\mathbf{A}}{\boxed{\begin{tikzcd} A \arrow[r, "F^A Fa"] \arrow[dr, "\forall a"'] & \operatorname{tgt} F^A Fa \arrow[d, dashed, "\exists! u"] \\ & A' \end{tikzcd}}} \quad \overset{}{\underset{F}{\longmapsto}} \quad \underset{\mathbf{B}}{\boxed{\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[dr, "Fa"'] & FA' \arrow[d, equal] \\ & FA' \end{tikzcd}}}$$

In particular, every discrete opfibration is a split opfibration.

CHAPTER 3

Proxy pullbacks

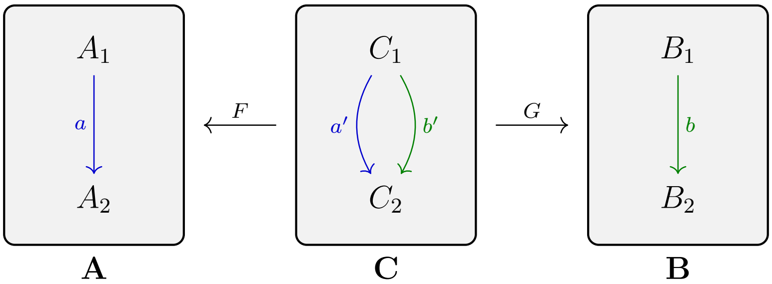
From the work of Chollet et al. [8], we know that $\mathcal{L}ens$ has equalisers and a terminal object. Were $\mathcal{L}ens$ to also have all binary products, then it would have all finite limits. Unfortunately, the product of the category $\mathbf{2}$ with itself does not exist in $\mathcal{L}ens$ (we will prove this shortly), so $\mathcal{L}ens$ does not have all binary products, nor all pullbacks. Despite this, there is a canonical way to produce a commuting square on any cospan of lenses and these commuting squares turn out to satisfy analogues of many of the properties of real pullbacks. This construction first appeared in Johnson and Rosebrugh's characterisation of symmetric delta lenses as equivalence classes of spans of asymmetric delta lenses (which we are merely calling lenses) [17]; they use it to define the composition of these spans. Johnson and Rosebrugh called it the *"pullback"* [17] of lenses, with the inverted quotes used intentionally to signal that it is not a real pullback. It has also been called the *imported pullback* by Chollet et al. [8]. We will adopt the name *proxy pullback* from Bumpus and Kocsis [7]. This name is justified as the dual of Bumpus and Kocsis' property SC2 for *proxy pushouts* does indeed hold for proxy pullbacks in $\mathcal{L}ens$; actually, a more general property holds, as we shall see in Proposition 3.14. Other pullback-like constructions include Böhm's pullback relative to a class of spans [6], and Simpson's conditional independent product [29].

Johnson and Rosebrugh's definition of their "pullback" of lenses singles out a specific lens span on each lens cospan. We will give a new alternative definition in which a proxy pullback is a lens square that satisfies a certain property. With our new definition, each lens cospan may have several proxy pullbacks, however there is always a unique span isomorphism between any two proxy pullbacks of the same lens cospan, as we shall see in Corollary 3.15. Essential to our definition of proxy pullback is the notion of a *compatible lens square*, which itself derives from the more primitive notion of a *compatible square of functors and cofunctors*. This is where our chapter begins. Later, after defining the proxy pullback, we will show that for any chosen pullback of the get functors of a lens cospan, there is a unique proxy pullback of the lens cospan whose get functors are that chosen pullback. This last result is similar to a limit creation property, where proxy pullbacks are, in some sense, created from real pullbacks in $\mathcal{C}at$ by the functor $\mathcal{G}\colon \mathcal{L}ens \to \mathcal{C}at$. We conclude with several other pullback-like properties of proxy pullbacks, including an analogue of the well-known *pullback pasting lemma*.

Before proceeding, on the next page we will prove the claim above that the product of the category $\mathbf{2}$ with itself does not exist in $\mathcal{L}ens$. The proof uses Chollet et al.'s observation that the forgetful functor $\mathcal{G}\colon \mathcal{L}ens \to \mathcal{C}at$ reflects monos [8].

Proposition 3.1. *The product of the category* **2** *with itself does not exist in* $\mathcal{L}ens$.

Proof. Consider the lens span



between the isomorphic copies **A** and **B** of **2**, where F and G are given explicitly by

$$Fa' = a = Fb' \qquad\qquad F^{C_1}a = a'$$
$$Ga' = b = Gb' \qquad\qquad G^{C_1}b = b'.$$

Assume that **A** and **B** have a product $\mathbf{A} \times \mathbf{B}$ in $\mathcal{L}ens$, and let $P_1 \colon \mathbf{A} \times \mathbf{B} \to \mathbf{A}$ and $P_2 \colon \mathbf{A} \times \mathbf{B} \to \mathbf{B}$ be the projection lenses. Let $L \colon \mathbf{C} \to \mathbf{A} \times \mathbf{B}$ be the unique lens such that $P_1 \circ L = F$ and $P_2 \circ L = G$. We claim that L is a monic lens. We have $LC_1 \neq LC_2$ as

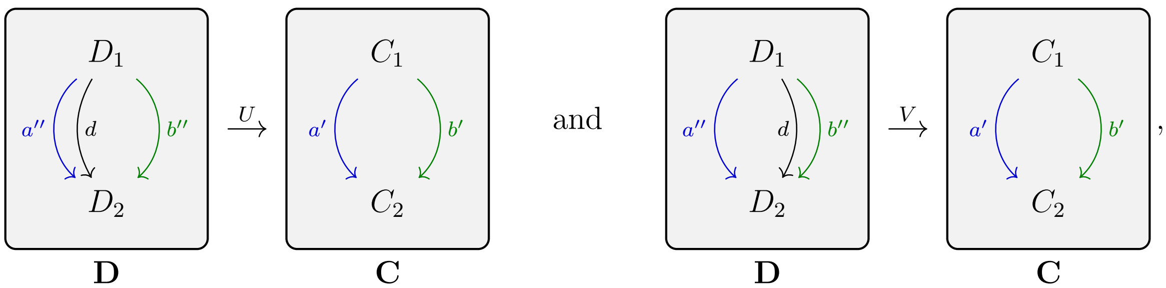
$$P_1 LC_1 = FC_1 = A_1 \neq A_2 = FC_2 = P_1 LC_2.$$

Hence L is injective on objects, and La' and Lb' are both neither $L\,\mathrm{id}_{C_1}$ nor $L\,\mathrm{id}_{C_2}$. Now

$$L^{C_1}La' = L^{C_1}LF^{C_1}a = L^{C_1}LL^{C_1}P_1{}^{LC_1}a = L^{C_1}P_1{}^{LC_1}a = F^{C_1}a = a',$$

and similarly $L^{C_1}Lb' = b'$, so we have $L^{C_1}La' = a' \neq b' = L^{C_1}Lb'$, and thus $La' \neq Lb'$. Hence L is injective on morphisms. As the get functor of L is injective on objects and on morphisms, it is monic. As $\mathcal{G}$ reflects monos, L must itself be a monic lens.

Consider now also the lenses



where U and V are given explicitly by

$$Ua'' = a' = Va'' \qquad Ub'' = b' = Vb'' \qquad Ud = a'$$
$$U^{D_1}a' = a'' = V^{D_1}a' \qquad U^{D_1}b' = b'' = V^{D_1}b' \qquad Vd = b'.$$

One may check that $P_1 \circ L \circ U = P_1 \circ L \circ V$ and similarly $P_2 \circ L \circ U = P_2 \circ L \circ V$. As product projection spans are jointly monic, and L is also monic, we get the contradiction $U = V$. $\square$

3.1. Compatible squares of functors and cofunctors

The notion of compatible square arises quite naturally when considering the interaction between functors and cofunctors. Not only is it central to our definition of proxy pullback, but it is actually implicit in the definition of lens itself.

Definition 3.2. A *compatible square of functors and cofunctors* is a commuting mixed diagram

$$\begin{tikzcd} \mathbf{D} \arrow[r, "\overline{F}"] \arrow[d, "\overline{G}"'] & \mathbf{B} \arrow[d, "G"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{C} \end{tikzcd}$$

such that for all $D \in |\mathbf{D}|$ and all $a \in \mathbf{A}(\overline{G}D, *)$,

$$\overline{F}\overline{G}^D a = G^{\overline{F}D} F a.$$

Remark 3.3. A lens $\mathbf{A} \to \mathbf{B}$ is exactly a compatible square

$$\begin{tikzcd} \mathbf{A} \arrow[r, "G"] \arrow[d, "P"'] & \mathbf{B} \arrow[d, equal] \\ \mathbf{B} \arrow[r, equal] & \mathbf{B} \end{tikzcd}$$

of functors and cofunctors. The functor G is the get functor of the lens and the cofunctor P is the put cofunctor of the lens. The compatibility condition says that the get functor and put cofunctor both have the same object function and that they satisfy the PutGet axiom. A lens as above is a discrete opfibration if and only if the mixed diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, equal] \arrow[d, equal] & \mathbf{A} \arrow[d, "P"] \\ \mathbf{A} \arrow[r, "G"'] & \mathbf{B} \end{tikzcd}$$

is also a compatible square; compatibility of this square encodes the GetPut axiom.

There is a notion of transformation between two diagrams in $\mathcal{C}at$ of the same shape that is similar to that of a natural transformation, except that its components are cofunctors rather than functors, and its squares are compatible squares rather than commuting squares; we will call such a transformation a *compatible-cofunctor transformation* between the diagrams. Shortly, in Theorem 3.6, we will see that for a pair of diagrams in $\mathcal{C}at$ of the same shape and a limit cone on each diagram, if there is a *compatible-cofunctor* transformation between the diagrams then there is a unique comparison *cofunctor* between the apices of the limit cones that forms compatible squares with the components of the limit cones and the compatible transformation. This mirrors the fact that there is a unique comparison *functor* between the apices of the limit cones when there is a *natural* transformation between the diagrams.

Our result about compatible-cofunctor transformations mentioned in the previous paragraph is best stated with respect to a compatible-square analogue of the category $\mathcal{C}at^{\mathbf{2}}$ of commuting squares of functors. In order for compatible squares to form the morphisms of a category, the pasting of two compatible squares ought to be a compatible square itself.

Lemma 3.4. *Horiztonal and vertical pastings of compatible squares of functors and cofunctors are also compatible squares of functors and cofunctors.*

Both pasting operations have compatible squares which act as identities. For horizontal pasting, these are the ones in which both functors are identity functors. For vertical pasting, these are the ones in which both cofunctors are identity cofunctors. The interchange law for these two pasting operations holds trivially.

Let $\mathcal{C}of\mathcal{S}q$ be the category whose objects are cofunctors and whose morphisms from a cofunctor $F\colon \mathbf{A}_1 \to \mathbf{A}_2$ to a cofunctor $G\colon \mathbf{B}_1 \to \mathbf{B}_2$ are the compatible squares of shape

$$\begin{tikzcd}
\mathbf{A}_1 \arrow[r] \arrow[d, "F"'] & \mathbf{B}_1 \arrow[d, "G"] \\
\mathbf{A}_2 \arrow[r] & \mathbf{B}_2
\end{tikzcd} .$$

Composition is given by horizontal pasting and identities are as described above. There are *source* and *target* functors $\mathcal{S}, \mathcal{T}\colon \mathcal{C}of\mathcal{S}q \to \mathcal{C}at$ that respectively map each cofunctor to its source and target category, and each compatible square to its top and bottom functor.

Remark 3.5. A reader familiar with double categories will have noticed that above we described various aspects of a *flat strict double category* of categories, functors, cofunctors and compatible squares. We will have more to say about this in Chapter 7.

THEOREM 3.6. *The functor* $\langle \mathcal{S}, \mathcal{T} \rangle\colon \mathcal{C}of\mathcal{S}q \to \mathcal{C}at \times \mathcal{C}at$ *creates limits.*

PROOF. Let $\mathcal{F}$ be a $\mathbf{J}$-shaped diagram in $\mathcal{C}of\mathcal{S}q$. Let $\mathcal{A}$ and $\mathcal{B}$ be the respective $\mathbf{J}$-shaped diagrams $\mathcal{S} \circ \mathcal{F}$ and $\mathcal{T} \circ \mathcal{F}$ in $\mathcal{C}at$. The image by $\mathcal{F}$ of a morphism $j\colon J_1 \to J_2$ of $\mathbf{J}$ in $\mathcal{C}of\mathcal{S}q$ can be written as the compatible square

$$\begin{tikzcd}
\mathcal{A}J_1 \arrow[r, "\mathcal{A}j"] \arrow[d, "\mathcal{F}J_1"'] & \mathcal{A}J_2 \arrow[d, "\mathcal{F}J_2"] \\
\mathcal{B}J_1 \arrow[r, "\mathcal{B}j"'] & \mathcal{B}J_2
\end{tikzcd} . \tag{1}$$

Let α and β be limit cones in $\mathcal{C}at$ on the respective diagrams $\mathcal{A}$ and $\mathcal{B}$ with respective apices $\mathbf{A}$ and $\mathbf{B}$. First, we will show that there is a unique cofunctor $F\colon \mathbf{A} \to \mathbf{B}$ such that

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\alpha_J"] \arrow[d, "F"'] & \mathcal{A}J \arrow[d, "\mathcal{F}J"] \\
\mathbf{B} \arrow[r, "\beta_J"'] & \mathcal{B}J
\end{tikzcd} . \tag{2}$$

is a compatible square for each $J \in |\mathbf{J}|$. Later, we will see that these compatible squares form the components of a limit cone in $\mathcal{C}of\mathcal{S}q$ on the the diagram $\mathcal{F}$.

We begin with uniqueness. Suppose that such a cofunctor $F\colon \mathbf{A} \to \mathbf{B}$ exists. Let $A \in |\mathbf{A}|$ and let $b \in \mathbf{B}(FA, *)$. For each $J \in |\mathbf{J}|$, we have $\beta_J FA = (\mathcal{F}J)\alpha_J A$ and $\alpha_J F^A b = (\mathcal{F}J)^{\alpha_J A}\beta_J b$ by compatibility of the square (2). Universality of the limit cones β and α then respectively determine the values of FA and $F^A b$.

We now prove existence. Define the object and lifting functions of F pointwise from the universal properties of the limit cones β and α as above. The cofunctor axioms PutId, PutPut and PutTgt for F follow from the same axioms for each of the cofunctors $\mathcal{F}J$, together with the universality of the limit cones α and β and the functoriality of their components. We see immediately from the definition of F that each of the squares (2) is a compatible square.

The compatible squares (2) do indeed form the components of a cone on the diagram $\mathcal{F}$. It remains to show that this cone is actually a limit cone. Consider another cone on the diagram $\mathcal{F}$ whose J-component is a compatible square depicted as the outer rectangle in the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, dashed, "U"] \arrow[d, "G"'] \arrow[rr, bend left, "\gamma_J"] & \mathbf{A} \arrow[r, "\alpha_J"] \arrow[d, "F"'] & \mathcal{A}J \arrow[d, "\mathcal{F}J"] \\
\mathbf{D} \arrow[r, dashed, "V"'] \arrow[rr, bend right, "\delta_J"'] & \mathbf{B} \arrow[r, "\beta_J"'] & \mathcal{B}J
\end{tikzcd} .$$

From the universal properties of the limit cones α and β, there are unique functors $U\colon \mathbf{C} \to \mathbf{A}$ and $V\colon \mathbf{D} \to \mathbf{B}$ such that, for each $J \in |\mathbf{J}|$, the top and bottom triangles in the diagram commute. It suffices to show that the left-hand square in the diagram is a compatible square.

Let $C \in |\mathbf{C}|$ and $d \in \mathbf{D}(GC, *)$. For each $J \in |\mathbf{J}|$, we have

$$\beta_J FUC = (\mathcal{F}J)\alpha_J UC = (\mathcal{F}J)\gamma_J C = \delta_J GC = \beta_J VGC$$

and also

$$\alpha_J U G^C d = \gamma_J G^C d = (\mathcal{F}J)^{\gamma_J C}\delta_J d = \alpha_J F^{UC} V d.$$

From the universal properties of the limit cones β and α, we may deduce, respectively, that $FUC = VGC$ and that $UG^C d = F^{UC} V d$. □

All major results about proxy pullbacks in this chapter may be thought of as following, ultimately, from Lemma 3.4 and Theorem 3.6.

3.2. Compatible squares of lenses and the proxy pullback

In the previous section, we defined the notion of compatibility for mixed diagrams of a special shape—squares with one pair of opposite sides being functors and the other pair of opposite sides being cofunctors. We may think of this notion of compatibility as a generalised notion of commutativity for mixed diagrams of this special shape. There are eight mixed diagrams underlying a lens square; for each lens in the square, we must choose whether to keep its get functor or its put cofunctor. The usual notion of commutativity makes sense for two of these underlying mixed diagrams, and our notion of compatible square of functors and cofunctors makes sense for two more of these underlying mixed diagrams. A *compatible lens square* will be one in which all four of these underlying mixed diagrams commutes or is compatible as appropriate; this is more restrictive than merely asking that the lens square commute.

Definition 3.7. A *compatible lens square* is a commuting lens square

$$
\begin{tikzcd}
\mathbf{D} \arrow[r, "\overline{F}"] \arrow[d, "\overline{G}"'] & \mathbf{B} \arrow[d, "G"] \\
\mathbf{A} \arrow[r, "F"'] & \mathbf{C}
\end{tikzcd}
\tag{3}
$$

such that the mixed diagrams

$$
\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathcal{G}\overline{F}"] \arrow[d, "\mathcal{P}\overline{G}"'] & \mathbf{B} \arrow[d, "\mathcal{P}G"] \\
\mathbf{A} \arrow[r, "\mathcal{G}F"'] & \mathbf{C}
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathcal{P}\overline{F}"] \arrow[d, "\mathcal{G}\overline{G}"'] & \mathbf{B} \arrow[d, "\mathcal{G}G"] \\
\mathbf{A} \arrow[r, "\mathcal{P}F"'] & \mathbf{C}
\end{tikzcd}
$$

are compatible squares of functors and cofunctors. We also say that the lens span $(\overline{G}, \overline{F})$ is *compatible with* the lens cospan (F, G).

Explicitly, the compatibility conditions above require that the equations

$$\overline{F}\overline{G}^D a = G^{\overline{F}D} F a \qquad \text{and} \qquad \overline{G}\overline{F}^D b = F^{\overline{G}D} G b$$

hold whenever they are defined.

Proposition 3.8. *The pasting of two compatible lens squares is a compatible lens square.*

Proof. This follows directly from Lemma 3.4. □

Proposition 3.9. *Every commuting lens square for which one leg of the cospan is a discrete opfibration is a compatible lens square.*

To understand the following proof of the above proposition, it will be helpful to recall from Remark 3.3 the compatible squares of functors and cofunctors that encode the PutGet axiom of a lens and the GetPut axiom of a discrete opfibration. By Lemma 3.4, we may horizontally and vertically paste such compatible squares together along common sides and the result is always a compatible square. There is no ambiguity as to which compatible square of functors and cofunctors is represented by a given pasting diagram of such compatible squares as each such compatible square is completely determined by its boundary.

Proof. Consider a commuting lens square (3). Without loss of generality, suppose that the lens F is a discrete opfibration. Each square in the pasting diagrams

$$
\begin{tikzcd}
\mathbf{D} \arrow[r, equal] \arrow[d, "\mathcal{P}\overline{G}"'] & \mathbf{D} \arrow[r, equal] \arrow[d, "\mathcal{P}\overline{G}"] & \mathbf{D} \arrow[r, "\mathcal{G}\overline{F}"] \arrow[d, "\mathcal{P}\overline{F}"'] & \mathbf{B} \arrow[d, equal] \\
\mathbf{A} \arrow[r, equal] \arrow[d, equal] & \mathbf{A} \arrow[d, "\mathcal{P}F"] & \mathbf{B} \arrow[r, equal] \arrow[d, "\mathcal{P}G"'] & \mathbf{B} \arrow[d, "\mathcal{P}G"] \\
\mathbf{A} \arrow[r, "\mathcal{G}F"'] & \mathbf{C} \arrow[r, equal] & \mathbf{C} \arrow[r, equal] & \mathbf{C}
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathcal{G}\overline{G}"] \arrow[d, equal] & \mathbf{A} \arrow[r, equal] \arrow[d, equal] & \mathbf{A} \arrow[d, "\mathcal{P}F"] \\
\mathbf{D} \arrow[r, "\mathcal{G}\overline{G}"'] \arrow[d, equal] & \mathbf{A} \arrow[r, "\mathcal{G}F"'] & \mathbf{C} \arrow[d, equal] \\
\mathbf{D} \arrow[r, "\mathcal{G}\overline{F}"] \arrow[d, "\mathcal{P}\overline{F}"'] & \mathbf{B} \arrow[r, "\mathcal{G}G"] \arrow[d, equal] & \mathbf{C} \arrow[d, equal] \\
\mathbf{B} \arrow[r, equal] & \mathbf{B} \arrow[r, "\mathcal{G}G"'] & \mathbf{C}
\end{tikzcd}
$$

is a compatible square of functors and cofunctors. For example, in the left pasting diagram, the lower left square encodes the GetPut axiom of the discrete opfibration F, the upper right one encodes the PutGet axiom of the lens $\overline{F}$, and the middle one is the horizontal pasting identity square on the cofunctor $\mathcal{P}F \circ \mathcal{P}\overline{G} = \mathcal{P}G \circ \mathcal{P}\overline{F}$. By Lemma 3.4, the boundaries of these pasting diagrams are also compatible squares of functors and cofunctors. □

Remark 3.10. As all identity lenses are discrete opfibrations, the above proposition implies that every commuting lens triangle becomes a compatible lens square by inserting an identity lens into the triangle in the appropriate place.

Example 3.11. Not all commuting lens squares are compatible lens squares. In the following, we use the previously introduced notation for **2** and **1**. Consider the commuting square

$$\begin{tikzcd} \mathbf{2} \arrow[r, equal, "\overline{G}"] \arrow[d, equal, "\overline{F}"'] & \mathbf{2} \arrow[d, "G"] \\ \mathbf{2} \arrow[r, "F"'] & \mathbf{1} \end{tikzcd}$$

in $\mathcal{L}ens$. The lenses F and G are both the same unique lens to the terminal category, and the lenses $\overline{F}$ and $\overline{G}$ are both the same identity lens on the category **2**. The square is not compatible as $G^0 Fu = G^0 \operatorname{id}_0 = \operatorname{id}_0$ whilst $\overline{G}\,\overline{F}^0 u = \overline{G}u = u$.

Definition 3.12. A *proxy-pullback square* is a compatible lens square sent by $\mathcal{G}$ to a pullback square in $\mathcal{C}at$. A *proxy pullback* of a lens cospan is a lens span forming a proxy-pullback square with the cospan. In diagrams, we will mark proxy-pullback squares with PPB. Additionally,

- a *proxy kernel pair* of a lens is a proxy pullback of the cospan with both legs the lens, and
- a *proxy product* is a proxy pullback of a cospan whose apex is the terminal category.

For a lens square (3) whose get functors form a pullback square in $\mathcal{C}at$, the universal property of the pullback gives the following equivalent characterisation of the compatibility conditions: for all $D \in |\mathbf{D}|$, all $a \in \mathbf{A}(\overline{G}D, *)$ and all $b \in \mathbf{B}(\overline{F}D, *)$,

$$\overline{F}\,\overline{G}^D a = G^{\overline{F}D} Fa \iff \begin{tikzcd} \mathbf{2} \arrow[dr, "\overline{G}^D a"] \arrow[drr, bend left, "G^{\overline{F}D}Fa"] \arrow[ddr, bend right, "a"'] & & \\ & \mathbf{D} \arrow[r, "\mathcal{G}\overline{F}"] \arrow[d, "\mathcal{G}\overline{G}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B} \arrow[d, "\mathcal{G}G"] \\ & \mathbf{A} \arrow[r, "\mathcal{G}F"'] & \mathbf{C} \end{tikzcd} \text{ commutes,}$$

$$\overline{G}\,\overline{F}^D b = F^{\overline{G}D} Gb \iff \begin{tikzcd} \mathbf{2} \arrow[dr, "\overline{F}^D b"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "F^{\overline{G}D}Gb"'] & & \\ & \mathbf{D} \arrow[r, "\mathcal{G}\overline{F}"] \arrow[d, "\mathcal{G}\overline{G}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B} \arrow[d, "\mathcal{G}G"] \\ & \mathbf{A} \arrow[r, "\mathcal{G}F"'] & \mathbf{C} \end{tikzcd} \text{ commutes.}$$

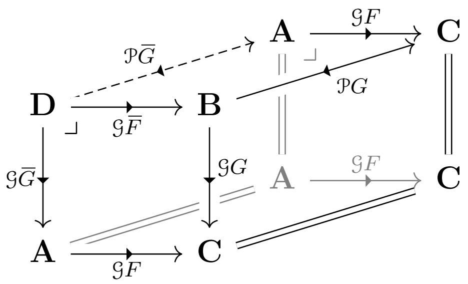
Actually, starting with a pullback in $\mathcal{C}at$ of the get functors of a lens span, the diagrams on the right in the above equivalences tell us how to define lifts on the pullback projection functors; it is easy to check that this turns these functors into lenses and that the resulting lens square is compatible. This is the creation-like result mentioned in the introduction to this chapter.

Proposition 3.13. *For each lens cospan, there is a unique proxy pullback of the cospan above each pullback of the get functors of the cospan.*

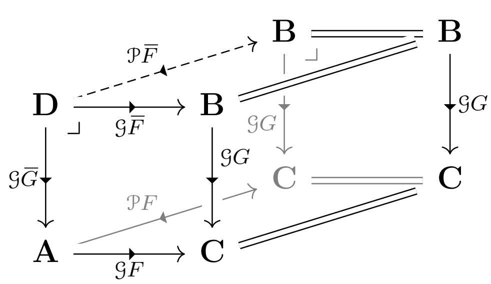
Rather than the nuts-and-bolts proof outlined above, we may instead deduce Proposition 3.13 from Theorem 3.6 with a high-level diagrammatic argument.

Proof. In summary, applying Theorem 3.6 to two suitably chosen compatible-cofunctor transformations between functor cospans gives us the put cofunctors of the lenses that we need to construct, as well as the compatible squares of functors and cofunctors needed for the constructed lens square to be a compatible one. Commutativity of the put cofunctors of this lens square comes from one final application of Theorem 3.6.

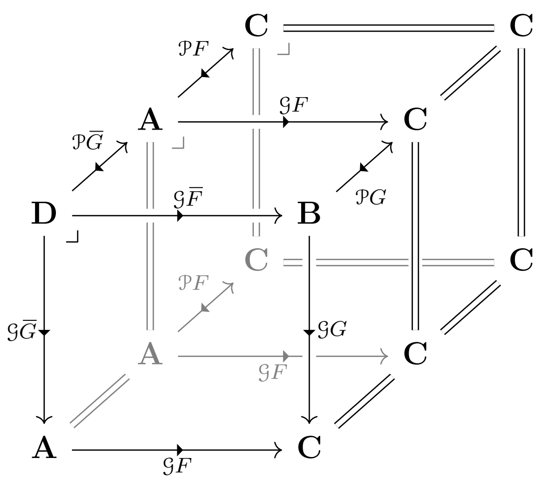
In detail, start with chosen pullbacks of $\mathcal{G}G$ and $\mathcal{G}F$ along each other in $\mathcal{C}at$. We denote them respectively by $\mathcal{G}\overline{G}$ and $\mathcal{G}\overline{F}$ as we will shortly construct lenses $\overline{G}$ and $\overline{F}$ with these functors as their get functors. By Theorem 3.6, as the bottom and right-hand faces of the cubes



and



are compatible squares of functors and cofunctors, there are unique cofunctors $\mathcal{P}\overline{G}$ and $\mathcal{P}\overline{F}$ as depicted such that the top and left-hand faces of the cubes are also compatible squares of functors and cofunctors. One of these compatible squares tells us that $\mathcal{G}\overline{G}$ and $\mathcal{P}\overline{G}$ form a lens $\overline{G}$, another that $\mathcal{G}\overline{F}$ and $\mathcal{P}\overline{F}$ form a lens $\overline{F}$, and the remaining two that the constructed lens square is compatible, so long as it also commutes. As the bottom and right-hand faces of the cube



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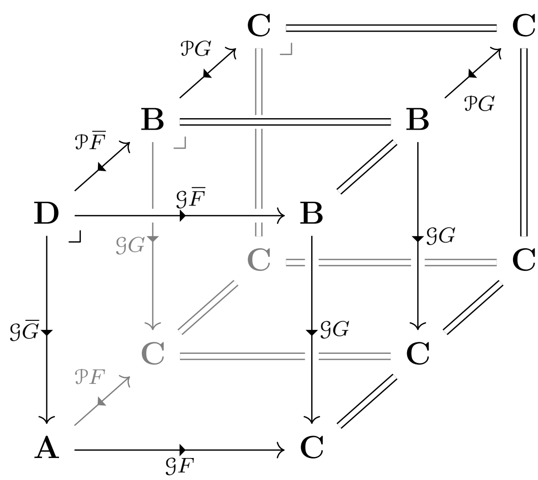


are compatible squares of functors and cofunctors, by Theorem 3.6, there is a unique cofunctor along top-left edge of the cube that such that its top and left-hand faces are also compatible squares of functors and cofunctors. As $\mathcal{P}F \circ \mathcal{P}\overline{G}$ and $\mathcal{P}G \circ \mathcal{P}\overline{F}$ are both such a cofunctor, they are equal by uniqueness. The lens span $(\overline{G}, \overline{F})$ is thus a proxy pullback of the lens cospan (F, G) that lies above the chosen pullback $(\mathcal{G}\overline{G}, \mathcal{G}\overline{F})$ of the cospan $(\mathcal{G}F, \mathcal{G}G)$ in $\mathcal{C}at$; the uniqueness of $(\overline{G}, \overline{F})$ follows from the uniqueness of $\mathcal{P}\overline{G}$ and $\mathcal{P}\overline{F}$. $\square$

Many of the pullback-like properties of the proxy pullback are consequences of the following proposition, which is a proxy-pullback analogue of Theorem 3.6.

Proposition 3.14. *Consider the diagram*

$$
\begin{tikzcd}
& \mathbf{D}' \arrow[rr, "\overline{F}'"] \arrow[dd, "\overline{G}'"'] & & \mathbf{B}' \arrow[dd, "G'"] \\
\mathbf{D} \arrow[ur, dashed, "T_D"] \arrow[rr, "\overline{F}"'] \arrow[dd, "\overline{G}"'] & & \mathbf{B} \arrow[ur, "T_B"'] \arrow[dd, "G"] \arrow[ur, phantom, "\mathsf{PPB}"] & \\
& \mathbf{A}' \arrow[rr, "F'"] & & \mathbf{C}' \\
\mathbf{A} \arrow[ur, "T_A"] \arrow[rr, "F"'] \arrow[uurr, phantom, "\mathsf{PPB}"] & & \mathbf{C} \arrow[ur, "T_C"'] &
\end{tikzcd}
\tag{4}
$$

in $\mathcal{L}ens$. If the bottom and right-hand faces of (4) are compatible lens squares, then

- *there is a unique functor $\mathcal{G}T_D \colon \mathbf{D} \to \mathbf{D}'$ such that the squares*

$$
\begin{tikzcd}
\mathbf{A} \arrow[d, "\mathcal{G}T_A"'] & \mathbf{D} \arrow[l, "\mathcal{G}\overline{G}"'] \arrow[d, "\mathcal{G}T_D"] \\
\mathbf{A}' & \mathbf{D}' \arrow[l, "\mathcal{G}\overline{G}'"]
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathcal{G}\overline{F}"] \arrow[d, "\mathcal{G}T_D"'] & \mathbf{B} \arrow[d, "\mathcal{G}T_B"] \\
\mathbf{D}' \arrow[r, "\mathcal{G}\overline{F}'"'] & \mathbf{B}'
\end{tikzcd}
$$

  in $\mathcal{C}at$ commute,

- *there is a unique cofunctor $\mathcal{P}T_D \colon \mathbf{D} \to \mathbf{D}'$ such that the mixed diagrams*

$$
\begin{tikzcd}
\mathbf{A} \arrow[d, "\mathcal{P}T_A"'] & \mathbf{D} \arrow[l, "\mathcal{G}\overline{G}"'] \arrow[d, "\mathcal{P}T_D"] \\
\mathbf{A}' & \mathbf{D}' \arrow[l, "\mathcal{G}\overline{G}'"]
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathcal{G}\overline{F}"] \arrow[d, "\mathcal{P}T_D"'] & \mathbf{B} \arrow[d, "\mathcal{P}T_B"] \\
\mathbf{D}' \arrow[r, "\mathcal{G}\overline{F}'"'] & \mathbf{B}'
\end{tikzcd}
$$

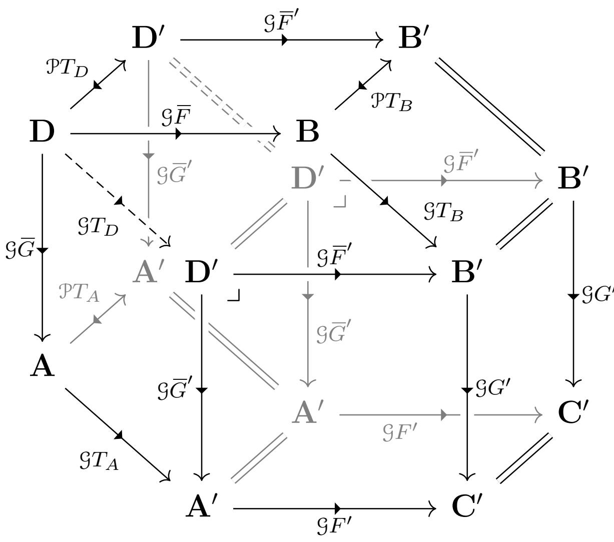
  are compatible squares of functors and cofunctors, and

- *the functor $\mathcal{G}T_D$ and cofunctor $\mathcal{P}T_D$ are actually the get functor and put cofunctor of a lens $T_D \colon \mathbf{D} \to \mathbf{D}'$ which makes the top and left-hand faces of (4) into compatible lens squares.*

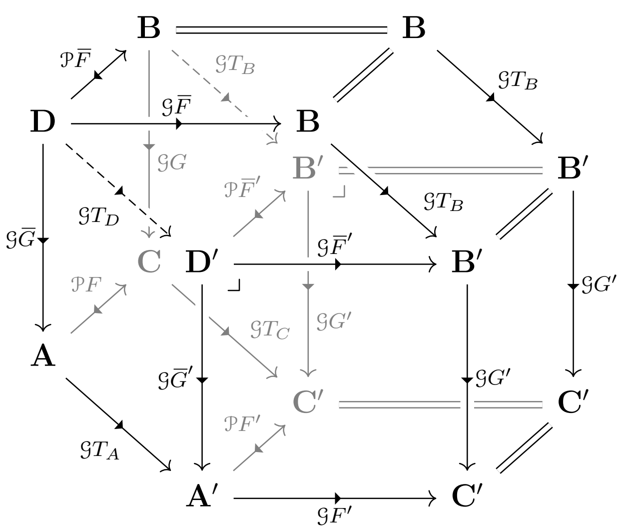
Proof. Consider the mixed diagrams

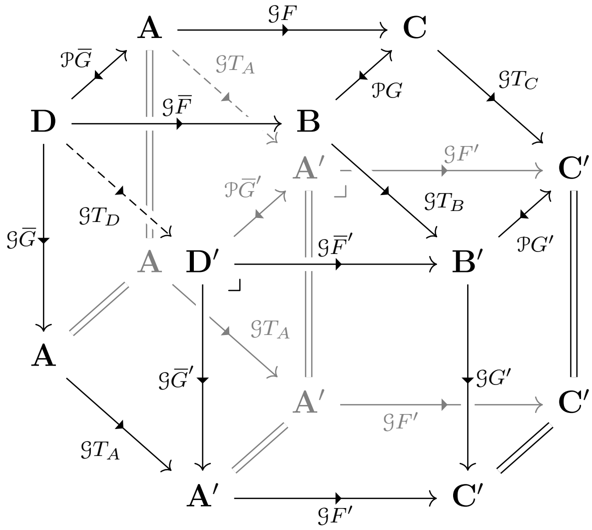
$$
\begin{tikzcd}
& \mathbf{D}' \arrow[rr, "\mathcal{G}\overline{F}'"] \arrow[dd, "\mathcal{G}\overline{G}'"'] & & \mathbf{B}' \arrow[dd, "\mathcal{G}G'"] \\
\mathbf{D} \arrow[ur, dashed, "\mathcal{G}T_D"] \arrow[rr, "\mathcal{G}\overline{F}"'] \arrow[dd, "\mathcal{G}\overline{G}"'] & & \mathbf{B} \arrow[ur, "\mathcal{G}T_B"'] \arrow[dd, "\mathcal{G}G"] & \\
& \mathbf{A}' \arrow[rr, "\mathcal{G}F'"] & & \mathbf{C}' \\
\mathbf{A} \arrow[ur, "\mathcal{G}T_A"] \arrow[rr, "\mathcal{G}F"'] & & \mathbf{C} \arrow[ur, "\mathcal{G}T_C"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathbf{D}' \arrow[rr, "\mathcal{G}\overline{F}'"] \arrow[dd, "\mathcal{G}\overline{G}'"'] & & \mathbf{B}' \arrow[dd, "\mathcal{G}G'"] \\
\mathbf{D} \arrow[ur, dashed, "\mathcal{P}T_D"] \arrow[rr, "\mathcal{G}\overline{F}"'] \arrow[dd, "\mathcal{G}\overline{G}"'] & & \mathbf{B} \arrow[ur, "\mathcal{P}T_B"'] \arrow[dd, "\mathcal{G}G"] & \\
& \mathbf{A}' \arrow[rr, "\mathcal{G}F'"] & & \mathbf{C}' \\
\mathbf{A} \arrow[ur, "\mathcal{P}T_A"] \arrow[rr, "\mathcal{G}F"'] & & \mathbf{C} \arrow[ur, "\mathcal{P}T_C"'] &
\end{tikzcd}
.
$$

Let $\mathcal{G}T_D$ be the unique comparison functor in the left-hand diagram from the universal property of the pullback, and let $\mathcal{P}T_D$ be the unique comparison cofunctor in the right-hand diagram from Theorem 3.6. Applying Theorem 3.6 to the diagram

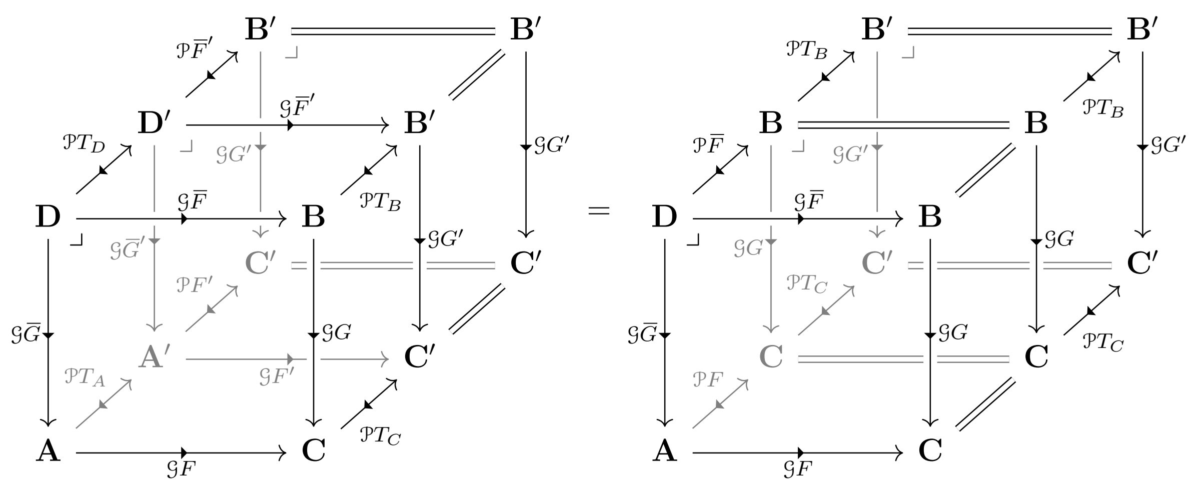


in $\mathcal{C}of\mathcal{S}q$, we see that $\mathcal{G}T_D$ and $\mathcal{P}T_D$ form a lens T_D. It remains to verify that the two lens squares involving T_D are compatible lens squares. Commutativity of the get functors of these squares comes from the top and left-hand faces of the cube defining $\mathcal{G}T_D$. The compatibility conditions involving $\mathcal{P}T_D$ come from the top and left-hand faces of the cube defining $\mathcal{P}T_D$. The compatibility conditions involving $\mathcal{G}T_D$ are obtained by applying Theorem 3.6 to the diagrams

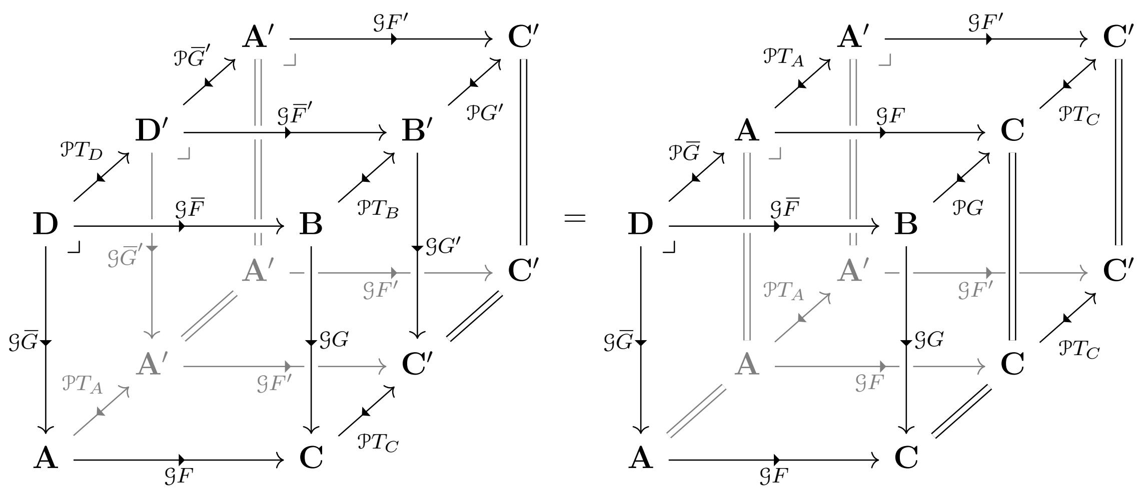




in $\mathcal{C}of\mathcal{S}q$. Finally, applying Theorem 3.6 to the cubes



and



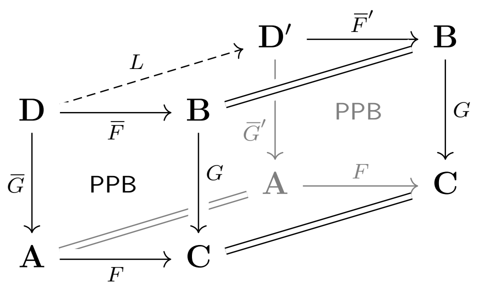
gives us the commutativity of the put cofunctors of the squares. $\square$

A *compatible-lens transformation* is a natural transformation between diagrams in $\mathcal{L}ens$ whose naturality squares are actually compatible lens squares. Just as we interpreted Theorem 3.6 in terms of compatible-cofunctor transformations, we may also interpret the above proposition in terms of compatible-lens transformations—a compatible-lens transformation between lens cospans extends uniquely to a compatible-lens transformation between chosen proxy-pullback squares on those cospans.

Corollary 3.15. *Proxy-pullback spans are unique up to unique span isomorphism.*

Proof. The proof proceeds similarly to a standard proof of the same result for pullbacks, using Proposition 3.14 as a substitute for the universal property of a pullback.

In detail, let $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ and $\mathbf{A} \xleftarrow{\overline{G}'} \mathbf{D}' \xrightarrow{\overline{F}'} \mathbf{B}$ be lens spans that are both proxy pullbacks of the same lens cospan $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$. The bottom and right-hand faces of the cube



in $\mathcal{L}ens$ are compatible lens squares, so by Proposition 3.14, there is a unique lens L as depicted such that the top and left-hand faces of the cube are also compatible lens squares. In particular, this means that L is a lens span morphism from $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ to $\mathbf{A} \xleftarrow{\overline{G}'} \mathbf{D}' \xrightarrow{\overline{F}'} \mathbf{B}$. Viewing the cube as instead going from the back face to the front face, we similarly obtain a lens span morphism L' from $\mathbf{A} \xleftarrow{\overline{G}'} \mathbf{D}' \xrightarrow{\overline{F}'} \mathbf{B}$ to $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$. Applying Proposition 3.14 to the cube

$$\begin{tikzcd}
 & \mathbf{D} \arrow[rr, "\overline{F}"] \arrow[dd, "\overline{G}"'] \arrow[ddrr, phantom, "\mathsf{PPB}"] & & \mathbf{B} \arrow[dd, "G"] \\
\mathbf{D} \arrow[ur, dashed, "L' \circ L"] \arrow[rr, "\overline{F}"'] \arrow[dd, "\overline{G}"'] \arrow[ddrr, phantom, "\mathsf{PPB}"] & & \mathbf{B} \arrow[ur, equal] \arrow[dd, "G"] & \\
 & \mathbf{A} \arrow[rr, "F"] & & \mathbf{C} \\
\mathbf{A} \arrow[ur, equal] \arrow[rr, "F"'] & & \mathbf{C} \arrow[ur, equal] &
\end{tikzcd},$$

we deduce that $L' \circ L = \mathrm{id}_{\mathbf{D}}$. We may similarly also deduce that $L \circ L' = \mathrm{id}_{\mathbf{D}'}$. ☐

3.3. Pullback-like properties of the proxy pullback

We conclude this chapter with several proxy analogues of well-known results about real products and pullbacks. Chollet et al. [8] already proved a few such results, including that

- the proxy product is a semi-cartesian symmetric monoidal product on the category $\mathcal{L}ens$ making $\mathcal{G} \colon \mathcal{L}ens \to \mathcal{C}at$ into a strong monoidal functor,
- proxy products distribute over coproducts,
- the category $\mathcal{L}ens$ is extensive[1].

It is also straightforward to check that identity lenses, isomorphisms, discrete opfibrations and split opfibrations are all proxy pullback stable.

The *pullback pasting lemma* says that a commuting square is a pullback square if and only if its pasting on the right with a pullback square is a pullback square.

Lemma 3.16 (Proxy-pullback pasting lemma)**.** *Consider the commuting diagram*

$$\begin{tikzcd}
\mathbf{D} \arrow[r, "T_D"] \arrow[d, "\overline{G}"'] & \mathbf{D}' \arrow[r, "\overline{F}'"] \arrow[d, "\overline{G}'"'] \arrow[dr, phantom, "\mathsf{PPB}"] & \mathbf{B}' \arrow[d, "G'"] \\
\mathbf{A} \arrow[r, "T_A"'] & \mathbf{A}' \arrow[r, "F'"'] & \mathbf{C}'
\end{tikzcd} \tag{5}$$

in $\mathcal{L}ens$ where the right-hand square is a proxy pullback. Then the left-hand square is a proxy pullback if and only if the outer rectangle is a proxy pullback and the mixed diagram

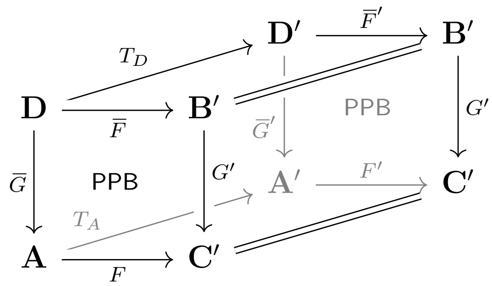
$$\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathcal{P}T_D"] \arrow[d, "\mathcal{G}\overline{G}"'] & \mathbf{D}' \arrow[d, "\mathcal{G}\overline{G}'"] \\
\mathbf{A} \arrow[r, "\mathcal{P}T_A"'] & \mathbf{A}'
\end{tikzcd} \tag{6}$$

is a compatible square of functors and cofunctors.

Proof. Suppose that the left-hand square of (5) is a proxy-pullback square. The outer rectangle of (5) is a compatible lens square by Proposition 3.8, and its get functors form a pullback square in $\mathcal{C}at$ by the pullback pasting lemma, so it is actually a proxy-pullback square.

[1] Coproduct inclusion lenses are discrete opfibrations so proxy pullbacks along them are real pullbacks.

Conversely, suppose that the outer rectangle of (5) is a proxy-pullback square and that (6) is a compatible square of functors and cofunctors. Let $F = F' \circ T_A$ and $\overline{F} = \overline{F}' \circ T_D$, so that



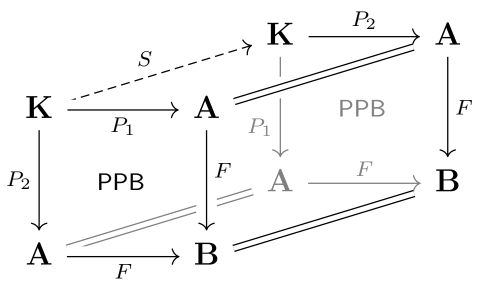
(7)

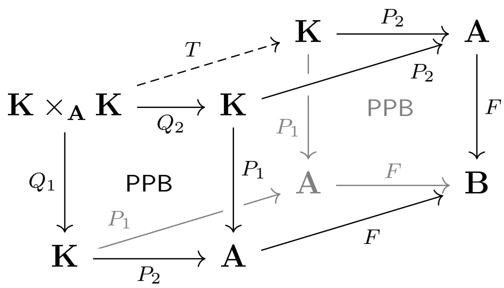
is a commuting diagram in $\mathcal{L}ens$. The bottom (Remark 3.10) and right-hand faces of (7) are compatible lens squares, so Proposition 3.14 applies. From Remark 3.10 and our assumption that (6) is a compatible square of functors and cofunctors, T_D is the unique lens given by Proposition 3.14, and so the top and left-hand faces of (7) are also compatible lens squares. The left-hand square in (5) is thus a compatible lens square, and its get functors form a pullback square in $\mathfrak{C}at$ by the pullback pasting lemma, so it is actually a proxy-pullback square. $\square$

Recall that an internal equivalence relation on an object X is a jointly monic span from X to X equipped with reflexivity, symmetry, and transitivity morphisms, and that every kernel pair is canonically an internal equivalence relation [See, for example, 21, Definition 1.3.6].

Corollary 3.17. *All proxy kernel pairs have symmetry and transitivity lenses.*

Proof. Let $P_1, P_2 \colon \mathbf{K} \to \mathbf{A}$ be a proxy kernel pair of a lens $F \colon \mathbf{A} \to \mathbf{B}$. The symmetry lens $S \colon \mathbf{K} \to \mathbf{K}$ and transitivity lens $T \colon \mathbf{K} \times_{\mathbf{A}} \mathbf{K} \to \mathbf{K}$ are constructed as in the diagrams





using Proposition 3.14, where the bottom and right-hand faces of the right-hand cube are compatible lens squares because they are proxy-pullback squares. $\square$

Remark 3.18. The proxy kernel pair of a lens F has a reflexivity lens if and only if F is a discrete opfibration; in this case, it is a real kernel pair.

In later chapters, we will see further parallels between proxy pullbacks and real pullbacks. This includes how they interact with monos, epis and image factorisations.

CHAPTER 4

Universal properties of the proxy pullback

Although the proxy pullback of a lens cospan is not in general a real pullback of the cospan, it is a real pullback when, for example, one of the legs of the cospan is a discrete opfibration [8]. The goal of this chapter is to better understand when a proxy pullback in $\mathcal{L}ens$ is a real pullback. We actually consider the more general question:

When does a lens span forming a commuting square with a lens cospan have a unique comparison lens to the proxy pullback of the cospan?

One way to approach our question, at least initially, is to look for lens span properties that are possessed by proxy pullback lens spans and that are preserved by precomposition with lenses. With respect to our question, a necessary condition for the existence of such a comparison lens is that the lens span possess these properties. We will consider two such properties. The first is compatibility with the cospan. The second is a new property of lens spans that we call *independence*, which is itself defined in terms of another new property of lens spans that we call *sync minimality*. Actually, as we will see later in the chapter, if the proxy-pullback span is itself sync minimal, then the possession of these two properties by the lens span is also a sufficient condition for the existence of such a comparison lens.

A natural next step is then to determine whether the sync minimality of the proxy-pullback span is itself also a necessary condition for the existence of such a comparison lens. This is not in general true, however it is actually a necessary condition for the *simultaneous* existence of a comparison lens to the proxy pullback from all independent lens spans that are compatible with the cospan. Stated differently, if a proxy-pullback span of a lens cospan is terminal amongst the independent spans that are compatible with the cospan, then the proxy-pullback span is necessarily sync minimal.

From the results mentioned above, one answer to our question is that the proxy pullback is a real pullback if and only if

- the proxy-pullback span is sync minimal, and
- every lens span that forms a commuting square with the cospan is independent and also compatible with the cospan.

Although this is a complete characterisation, it is somewhat unsatisfactory, as it is not expressed in terms of properties of the cospan that are easily checked. There is, however, such a satisfactory characterisation for lens cospans whose apex is the terminal category. Indeed, a proxy product of two categories is a real product if and only if at least one of the two categories is a discrete category. Finding such a satisfactory characterisation for general lens cospans is ongoing work.

4.1. Sync-minimal and independent lens spans

As was explained in the introduction to this chapter, the new notions of *sync minimality* and *independence* of lens spans give various necessary and sufficient conditions for the existence of comparison lenses into proxy pullbacks. In this section, we merely introduce these notions, delaying the development of their theory to when it is needed later in the chapter.

Johnson and Rosebrugh [17] proposed that we regard a lens span $\mathbf{A} \xleftarrow{F} \mathbf{C} \xrightarrow{G} \mathbf{B}$ as a synchronisation protocol between the systems represented by the categories $\mathbf{A}$ and $\mathbf{B}$. From this perspective, the category $\mathbf{C}$ has the sole purpose of coordinating the propagation to $\mathbf{B}$ of transitions that occur in $\mathbf{A}$ and vice versa. As transitions always originate in $\mathbf{A}$ or $\mathbf{B}$, there may be morphisms in $\mathbf{C}$ that are never used—these are the ones that are not composites of a sequence of morphisms that are all lifts along F or G. If there are no such extraneous morphisms in $\mathbf{C}$, we call the lens span *sync minimal*.

Definition 4.1. A lens span

$$\mathbf{A} \xleftarrow{\quad F \quad} \mathbf{C} \xrightarrow{\quad G \quad} \mathbf{B}$$

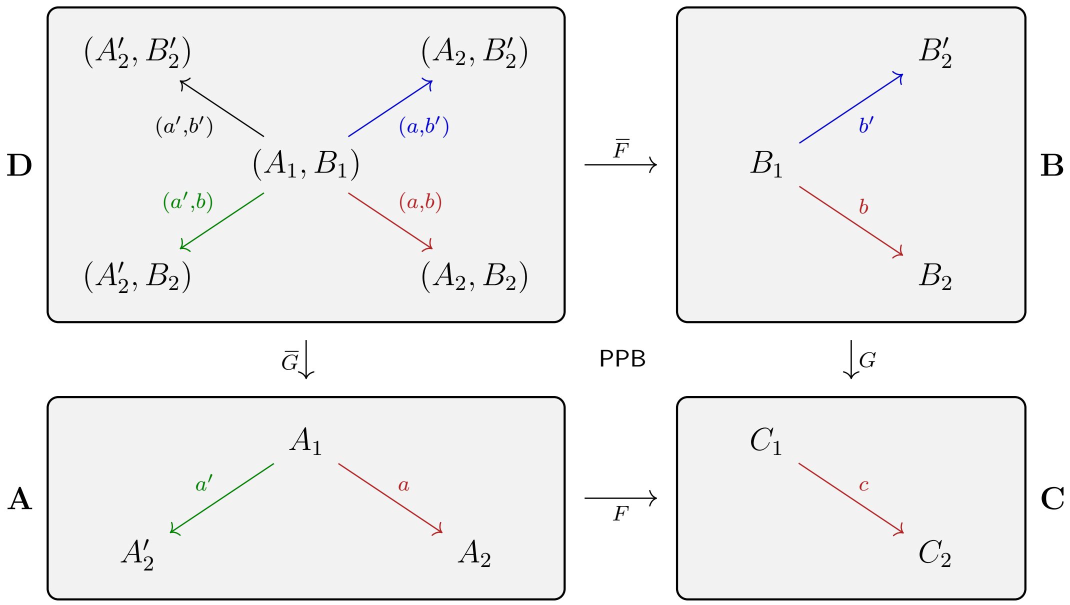
is *sync minimal* if each morphism in $\mathbf{C}$ is a composite of a sequence of morphisms

$$C_1 \xrightarrow{\quad c_1 \quad} C_2 \xrightarrow{\quad c_2 \quad} C_3 \;\cdots\; C_{n-1} \xrightarrow{\quad c_{n-1} \quad} C_n$$

that are all lifts along F or G, that is, for each k, either $c_k = F^{C_k} F c_k$ or $c_k = G^{C_k} G c_k$.

There are many sync-minimal lens spans, such as the lens span (F, G) in Proposition 3.1. However, not all proxy-pullback spans are sync minimal.

Example 4.2. Consider the proxy-pullback square depicted in the diagram below, where the lens lifts are indicated by the colouring of the morphisms.



The lens span $(\overline{G}, \overline{F})$ is not sync minimal as the morphism (a', b') is not a composite of lifts. Notice that removing (a', b') from $\mathbf{D}$ would make the span $(\overline{G}, \overline{F})$ sync minimal.

Starting with a lens span $\mathbf{A} \xleftarrow{F} \mathbf{C} \xrightarrow{G} \mathbf{B}$, by removing all morphisms in $\mathbf{C}$ that are not composites of a sequence of morphisms that are lifts along F or G, we obtain a sync-minimal lens span from $\mathbf{A}$ to $\mathbf{B}$ that encodes the same synchronisation protocol as (F, G). We call this sync-minimal lens span the *sync-minimal core* of (F, G) and denote it by $\mathcal{M}(F, G)$. Let $E_{(F,G)}$ denote the inclusion functor from the apex of $\mathcal{M}(F, G)$ to $\mathbf{C}$.

We are now ready to define the notion of *independence* for lens spans. It is similar to a jointly-monic condition, except only with respect to morphisms in the apex of the sync-minimal core of the span with the same source object. Defining independence with respect to the sync-minimal core is necessary for independence to be preserved by precomposition with lenses.

Definition 4.3. A lens span $\mathbf{A} \xleftarrow{F} \mathbf{C} \xrightarrow{G} \mathbf{B}$ is called *independent* if, for all morphisms c and c' in the apex of $\mathcal{M}(F, G)$ with the same source, whenever $Fc = Fc'$ and $Gc = Gc'$, also $c = c'$.

Remark 4.4. Simpson [29] defines the notion of *independent product* with respect to a chosen *independence structure*—a multicategory of multispans, called *independent* multispans, that satisfies certain additional properties. This is where our terminology for independent lens spans originates. We will have more to say about Simpson's independent products and local independent products at the end of this chapter.

The lens span $(\overline{G}, \overline{F})$ in Example 4.2 is independent. The lens span (F, G) in Proposition 3.1 is not independent.

Remark 4.5. Progress towards finding alternative characterisations of sync minimality and independence of a more category theoretic nature is discussed in Chapter 7.

4.2. Necessity of compatibility and independence

Proxy-pullback spans of a lens cospan are, by definition, compatible with the cospan. In this section, we will show that proxy-pullback spans are also independent, and that compatibility and independence of lens spans are preserved by precomposition with lenses. It follows that whenever a lens span that commutes with a lens cospan has a comparison lens to the proxy pullback of the cospan, the span is necessarily independent and compatible with the cospan. This is one of the claims made in the introduction to this chapter.

Proposition 4.6. *All proxy-pullback spans are independent.*

PROOF. Let $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ be a proxy pullback of some lens cospan. For all $D \in |\mathbf{D}|$, and all $d, d' \in \mathbf{D}(D, *)$, if $\overline{F}d = \overline{F}d'$ and $\overline{G}d = \overline{G}d'$ then $d = d'$ by the universal property of the pullback in $\mathcal{C}at$ underlying the proxy pullback. In particular, this holds for those objects and morphisms in the apex of $\mathcal{M}(\overline{G}, \overline{F})$ as it is a subcategory of $\mathbf{D}$. $\square$

Proposition 4.7. *Consider the following diagram in $\mathcal{L}ens$, where $K_1 = K_2 \circ H$ and $J_1 = J_2 \circ H$.*

$$\begin{tikzcd}
 & \mathbf{E}_1 \arrow[d, "H"] \arrow[ddl, bend right, "K_1"'] \arrow[ddr, bend left, "J_1"] & \\
 & \mathbf{E}_2 \arrow[dl, "K_2"'] \arrow[dr, "J_2"] & \\
\mathbf{A} \arrow[dr, "F"'] & & \mathbf{B} \arrow[dl, "G"] \\
 & \mathbf{C} &
\end{tikzcd}$$

If the span (K_2, J_2) is compatible with the cospan (F, G), then so is the span (K_1, J_1).

Proof. Suppose that the span (K_2, J_2) is compatible with the cospan (F, G). Then

$$F \circ K_1 = F \circ K_2 \circ H = G \circ J_2 \circ H = G \circ J_1,$$

so the span (K_1, J_1) forms a commuting square with the cospan (F, G). For one of the compatibility conditions, we have

$$J_1 {K_1}^E a = J_2 H H^E {K_2}^{HE} a = J_2 {K_2}^{HE} a = G^{J_2 HE} F a = G^{J_1 E} F a.$$

The other compatibility condition holds similarly. □

Proposition 4.8. *Consider the following commuting diagram in $\mathcal{L}ens$.*

$$\begin{tikzcd}
 & \mathbf{C}_1 \arrow[dl, "F_1"'] \arrow[dr, "G_1"] \arrow[dd, "H"] & \\
\mathbf{A} & & \mathbf{B} \\
 & \mathbf{C}_2 \arrow[ul, "F_2"] \arrow[ur, "G_2"'] &
\end{tikzcd}$$

If the span (F_2, G_2) is independent, then the span (F_1, G_1) is also independent.

Proof. Suppose that c and c' are morphisms in the apex of $\mathcal{M}(F_1, G_1)$ with the same source object C such that $F_1 c = F_1 c'$ and $G_1 c = G_1 c'$. Then Hc and Hc' are morphisms in the apex of $\mathcal{M}(F_2, G_2)$ with the same source object HC such that $F_2 Hc = F_2 Hc'$ and $G_2 Hc = G_2 Hc'$. As (F_2, G_2) is independent, actually $Hc = Hc'$. But c and c' are both composites of lifts along $H \circ F_2$ and $H \circ G_2$, so they are both lifts along H, and thus $c = H^C Hc = H^C Hc' = c'$. □

Although compatibility and independence give necessary conditions for the existence of a comparison lens, these conditions are not sufficient ones.

Example 4.9. Consider again the proxy-pullback square in Example 4.2. The sync-minimal core $\mathcal{M}(\overline{G}, \overline{F})$ of $(\overline{G}, \overline{F})$ is obtained by removing the morphism (a', b') from $\mathbf{D}$. Although the span $\mathcal{M}(\overline{G}, \overline{F})$ is independent and compatible with the cospan (F, G), there is no comparison *lens* from it to the proxy-pullback span $(\overline{G}, \overline{F})$. Assume that such a comparison lens exists. Then, as the put cofunctor of the comparison lens commutes with the put cofunctors of the legs of both spans, all of the morphisms in the apex of $\mathcal{M}(\overline{G}, \overline{F})$ are necessarily lifts of the corresponding morphisms in $\mathbf{D}$. The PutGet axiom necessitates that the lift by such a comparison lens of the morphism (a', b') into the apex of $\mathcal{M}(\overline{G}, \overline{F})$ be distinct from the lifts of the other morphisms (a', b), (a, b') and (a, b), but such a distinct morphism does not exist.

4.3. Necessity and sufficiency of sync minimality

In the previous section we saw that if a lens span that commutes with a lens cospan has a comparison lens to the proxy pullback of the cospan, then the lens span is necessarily independent and compatible with the cospan. It turns out that if the proxy pullback is also sync-minimal, then these necessary conditions are also sufficient ones.

Proposition 4.10. *Consider the following commuting diagram in $\mathcal{L}ens$.*

$$\begin{tikzcd}
& \mathbf{E} \arrow[d, dashed] \arrow[ddl, bend right, "K"'] \arrow[ddr, bend left, "J"] & \\
& \mathbf{D} \arrow[dl, "\overline{G}"'] \arrow[dr, "\overline{F}"] & \\
\mathbf{A} \arrow[dr, "F"'] & \text{PPB} & \mathbf{B} \arrow[dl, "G"] \\
& \mathbf{C} &
\end{tikzcd}$$

Suppose that the span (K, J) is independent and is compatible with the cospan (F, G). If the span $(\overline{G}, \overline{F})$ is sync minimal, then there is a unique lens $\mathbf{E} \to \mathbf{D}$ such that the triangles commute.

Proposition 4.10 follows directly from a combination of the following two lemmas.

Lemma 4.11. *Using the notation established in Proposition 4.10, let L be the unique comparison functor from the span $(\mathcal{G}K, \mathcal{G}J)$ to the pullback span $(\mathcal{G}\overline{G}, \mathcal{G}\overline{F})$. Then the mixed diagrams*

$$\begin{tikzcd}
\mathbf{A} \arrow[d, equal] & \mathbf{E} \arrow[l, "\mathcal{P}K"'] \arrow[d, "L"] \\
\mathbf{A} & \mathbf{D} \arrow[l, "\mathcal{P}\overline{G}"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{E} \arrow[r, "\mathcal{P}J"] \arrow[d, "L"'] & \mathbf{B} \arrow[d, equal] \\
\mathbf{D} \arrow[r, "\mathcal{P}\overline{F}"'] & \mathbf{B}
\end{tikzcd}$$

are compatible squares of functors and cofunctors.

Proof. For each $E \in |\mathbf{E}|$ and each $a \in \mathbf{A}(KE, *)$, we have

$$LK^E a = \langle \overline{G}LK^E a,\, \overline{F}LK^E a\rangle = \langle KK^E a,\, JK^E a\rangle = \langle a,\, G^{JE}Fa\rangle = \langle a,\, G^{\overline{F}LE}Fa\rangle = \overline{G}^{LE}a,$$

and so the left-hand mixed diagram above is a compatible square. The right-hand mixed diagram above is also a compatible square by a similar argument. □

Lemma 4.12. *Consider the following commuting mixed diagram.*

$$\begin{tikzcd}
& \mathbf{C}_1 \arrow[dl, "F_1"'] \arrow[dr, "G_1"] \arrow[dd, "H"] & \\
\mathbf{A} & & \mathbf{B} \\
& \mathbf{C}_2 \arrow[ul, "F_2"] \arrow[ur, "G_2"'] &
\end{tikzcd}$$

Suppose additionally that the the span (F_1, G_1) is independent and that the mixed diagrams

$$\begin{tikzcd}
\mathbf{A} \arrow[d, equal] & \mathbf{E} \arrow[l, "\mathcal{P}F_1"'] \arrow[d, "H"] \\
\mathbf{A} & \mathbf{D} \arrow[l, "\mathcal{P}F_2"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{E} \arrow[r, "\mathcal{P}G_1"] \arrow[d, "H"'] & \mathbf{B} \arrow[d, equal] \\
\mathbf{D} \arrow[r, "\mathcal{P}G_2"'] & \mathbf{B}
\end{tikzcd}$$

are compatible squares of functors and cofunctors. If the span (F_2, G_2) *is sync minimal, then there is a unique lens structure on* H *such that the resulting diagram in* $\mathcal{L}ens$ *commutes.*

PROOF. Suppose that (F_2, G_2) is sync minimal. If there is is such a lens structure on H, then, for all $C \in |\mathbf{C}_1|$, all $a \in \mathbf{A}(F_1C, *)$ and all $b \in \mathbf{B}(G_1C, *)$, we necessarily have

$$H^C {F_2}^{HC} a = {F_1}^C a \qquad \text{and} \qquad H^C {G_2}^{HC} b = {G_1}^C b;$$

that is, the lifts by H of those morphisms of $\mathbf{C}_2$ that are lifts by F_2 and G_2 are determined by the lifts by F_1 and G_1. As (F_2, G_2) is sync minimal, each morphism of $\mathbf{C}_2$ is a composite of such lifts, and so the above equations and the PutPut axiom for H determine the lifts by H of all morphisms of $\mathbf{C}_2$. Such a lens structure on H is thus uniquely determined if it exists. Actually, by the independence of (F_1, G_1), this gives H well-defined lifts. The PutPut axiom is immediate from the definition of H, the PutId axiom follows from that of F_1 (or of G_1), and the PutGet axiom follows from the assumed compatible squares. □

Although the sync minimality of a proxy pullback of a lens cospan is sufficient for the existence of a comparison lens to the proxy-pullback span from an independent lens span that is compatible with the lens cospan, it is not in general necessary. For example, there is always a comparison lens from any proxy-pullback span to itself, namely, the identity lens on its apex. However, sync minimality is in fact necessary for there to be such comparison lenses simultaneously from all of the independent lens spans that are compatible with the lens cospan.

Proposition 4.13. *Consider the proxy-pullback square in* $\mathcal{L}ens$ *depicted below.*

$$\begin{tikzcd} \mathbf{D} \arrow[r, "\overline{F}"] \arrow[d, "\overline{G}"'] \arrow[dr, phantom, "\mathsf{PPB}"] & \mathbf{B} \arrow[d, "G"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{C} \end{tikzcd}$$

If the proxy-pullback span $(\overline{G}, \overline{F})$ *is terminal amongst the independent spans that are compatible with the cospan* (F, G)*, then the proxy-pullback span* $(\overline{G}, \overline{F})$ *is sync minimal.*

To prove this proposition, we will consider what happens when there is a comparison lens to a proxy pullback from its sync-minimal core.

Lemma 4.14. *Let* $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ *be a lens span. The functor* $E_{(\overline{G},\overline{F})}$ *has a lens structure if and only if it is the identity functor on* $\mathbf{D}$*, in which case* $(\overline{G}, \overline{F}) = \mathcal{M}(\overline{G}, \overline{F})$ *is sync minimal.*

PROOF. By construction, the functor $E_{(\overline{G},\overline{F})}$ is an identity function on objects and is a subset inclusion on morphisms. A lens that is surjective on objects is also surjective on morphisms [8]. Hence, if there is a lens with get functor $E_{(\overline{G},\overline{F})}$, then $E_{(\overline{G},\overline{F})}$ is surjective on morphisms, and thus is actually the identity functor. Conversely, if $E_{(\overline{G},\overline{F})}$ is the identity functor on $\mathbf{D}$, then the identity lens on $\mathbf{D}$ is a lens with get functor $E_{(\overline{G},\overline{F})}$. □

PROOF OF PROPOSITION 4.13. Suppose that $(\overline{G}, \overline{F})$ is terminal amongst the independent spans that are compatible with (F, G). As the span $(\overline{G}, \overline{F})$ is a proxy pullback, it is by definition compatible with the cospan (F, G), and it is independent by Proposition 4.6. The independence of the span $\mathcal{M}(\overline{G}, \overline{F})$ and its compatibility with the cospan (F, G) follows from these properties of the span $(\overline{G}, \overline{F})$; the former from the way that the sync-minimal core is defined, and the latter because independence is defined in terms of the sync-minimal core. By our assumption, there is thus a comparison lens H from the span $\mathcal{M}(\overline{G}, \overline{F})$ to the span $(\overline{G}, \overline{F})$. By the universal property of the pullback span $(\mathcal{G}\overline{G}, \mathcal{G}\overline{F})$ in $\mathcal{C}at$, the functors $\mathcal{G}H$ and $E_{(\overline{G},\overline{F})}$ are both the unique comparison functor from the span of get functors of $\mathcal{M}(\overline{G}, \overline{F})$ to the pullback span $(\mathcal{G}\overline{G}, \mathcal{G}\overline{F})$, and so they are necessarily equal. The result then follows by Lemma 4.14. $\square$

4.4. Proxy pullbacks of split opfibrations

In the remainder of this chapter, we unpack the results in the previous two sections for the proxy pullback of a lens cospan with additional known properties. In this section, we consider what happens when one of the legs of the cospan is a split opfibration.

Proposition 4.15. *A proxy-pullback span of a lens cospan with one leg a split opfibration is terminal amongst the independent lens spans that are compatible with the cospan.*

Proposition 4.15 follows directly from Proposition 4.10 and the following lemma.

Lemma 4.16. *Consider a proxy-pullback square*

$$\begin{tikzcd}
\mathbf{D} \arrow[r, "\overline{F}"] \arrow[d, "\overline{G}"'] \arrow[dr, phantom, "\mathsf{PPB}"] & \mathbf{B} \arrow[d, "G"] \\
\mathbf{A} \arrow[r, "F"'] & \mathbf{C}
\end{tikzcd} .$$

If F or G is a split opfibration then the lens span $(\overline{G}, \overline{F})$ is sync minimal.

Proof. Without loss of generality, suppose that F is a split opfibration. Let $d\colon D_1 \to D_2$ be a morphism in $\mathbf{D}$, and let $a = \overline{G}d\colon A_1 \to A_2$ and $b = \overline{F}d\colon B_1 \to B_2$. Let u be the unique comparison morphism from the F-opcartesian morphism $F^{A_1}Fa$ to a, as in the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "F^{A_1}Fa"] \arrow[dr, "a"'] & A_2' \arrow[d, dashed, "u"] \\
 & A_2
\end{tikzcd} .$$

Then $d = \langle a,\, b\rangle = \langle u,\, \mathrm{id}_{B_2}\rangle \circ \langle F^{A_1}Fa,\, b\rangle = \langle u,\, G^{B_2}Fu\rangle \circ \langle F^{A_1}Gb,\, b\rangle = \overline{G}^{\langle A_2',\, B_2\rangle}u \circ \overline{F}^{\langle A_1,\, B_1\rangle}b.$ $\square$

Remark 4.17. Split opfibrations are pullback stable, so in the proof above $\overline{F}$ is actually a split opfibration. However, lens spans with one leg a split opfibration are not in general sync minimal.

Shortly we will see that for a lens cospan with one leg a split opfibration, the independence condition for lens spans forming compatible squares with the cospan is equivalent to a simpler notion of independence, which we will call *split independence*.

Definition 4.18. A lens span $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ is called $\overline{F}$-*split independent* if for all $D_1 \in |\mathbf{D}|$, all $a_1 \colon \overline{G}D_1 = A_1 \to A_1'$ in $\mathbf{A}$, all $b \colon \overline{F}D_1 = B_2 \to B_2$ in $\mathbf{B}$, and all $a_2 \colon \operatorname{tgt} \overline{G}\overline{F}^{D_1}b = A_2 \to A_2'$, as shown in the diagram

$$\underset{\mathbf{A}}{\begin{tikzcd} A_1 \arrow[r, "\overline{G}\overline{F}^{D_1}b"] \arrow[d, "a_1"'] & A_2 \arrow[d, "a_2"] \\ A_1' \arrow[r, "\overline{G}\overline{F}^{D_1'}b"'] & A_2' \end{tikzcd}} \quad \xleftarrow{\overline{G}}\!\shortmid \quad \underset{\mathbf{D}}{\begin{tikzcd} D_1 \arrow[r, "\overline{F}^{D_1}b"] \arrow[dd, "\overline{G}^{D_1}a_1"'] & D_2 \arrow[d, "\overline{G}^{D_2}a_2"] \\ & D_2' \\ D_1' \arrow[r, "\overline{F}^{D_1'}b"'] & \overline{D_2'} \end{tikzcd}} \quad \overset{\overline{F}}{\mapsto} \quad \underset{\mathbf{B}}{\begin{tikzcd} B_1 \arrow[r, "b"] \arrow[d, equal, "\overline{F}\overline{G}^{D_1}a_1"'] & B_2 \arrow[d, equal, "\overline{F}\overline{G}^{D_2}a_2"] \\ B_1 \arrow[r, "b"'] & B_2 \end{tikzcd}}, \tag{1}$$

whenever the square in $\mathbf{A}$ commutes and $\overline{F}\overline{G}^{D_1}a_1 = \mathrm{id}_{B_1}$ and $\overline{F}\overline{G}^{D_2}a_2 = \mathrm{id}_{B_2}$, then also $D_2' = \overline{D_2'}$ and the resulting square in $\mathbf{D}$ commutes.

Proposition 4.19. *Consider a compatible lens square*

$$\begin{tikzcd} \mathbf{D} \arrow[r, "\overline{F}"] \arrow[d, "\overline{G}"'] & \mathbf{B} \arrow[d, "G"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{C} \end{tikzcd} .$$

If F is a split opfibration, then $(\overline{G}, \overline{F})$ is independent if and only if it is $\overline{F}$-split independent.

The *only if* direction follows directly from the definition of independence. Essential to the proof of the *if* direction is the following lemma.

Lemma 4.20. *Suppose that F is a split opfibration and $(\overline{G}, \overline{F})$ is $\overline{F}$-split independent. Then, each morphism $d \colon D_1 \to D_2$ in the apex of $\mathcal{M}(\overline{G}, \overline{F})$ has the factorisation*

$$\begin{tikzcd} D_1 \arrow[r, "\overline{F}^{D_1}\overline{F}d"] \arrow[dr, "d"'] & D_2' \arrow[d, "\overline{G}^{D_2'}u"] \\ & D_2 \end{tikzcd} \tag{2}$$

where $u \colon \overline{G}D_3 \to \overline{G}D_2$ comes from the universal property of the F-opcartesian morphism $F^{\overline{G}D_1}G\overline{F}d$, that is, u is the unique morphism of $\mathbf{A}$ for which $Fu = \mathrm{id}_{F\overline{G}D_2}$ and the diagram

$$\begin{tikzcd} \overline{G}D_1 \arrow[r, "F^{\overline{G}D_1}G\overline{F}d"] \arrow[dr, "\overline{G}d"'] & \overline{G}D_2' \arrow[d, "u"] \\ & \overline{G}D_2 \end{tikzcd} \tag{3}$$

commutes. In particular, the right leg of $\mathcal{M}(\overline{G}, \overline{F})$ is also a split opfibration.

We will return to prove the lemma shortly, but let us first finish the proof of the proposition.

Proof of *if* direction of Proposition 4.19. If F is a split opfibration and $(\overline{G}, \overline{F})$ is $\overline{F}$-split independent, then Lemma 4.20 implies that each morphism d in the apex of $\mathcal{M}(\overline{G}, \overline{F})$ is uniquely determined by the data $\overline{G}d$ and $\overline{F}d$. Indeed, from (2), d is a composite of morphisms expressed in terms of $\overline{F}d$ and u, and u itself is uniquely determined by the top side and hypotenuse of the triangle (3), which are themselves expressed in terms of $\overline{G}d$ and $\overline{F}d$. $\square$

Proof of Lemma 4.20. Recall that each morphism in the apex of $\mathcal{M}(\overline{G}, \overline{F})$ is a composite

$$D_1 \xrightarrow{d_1} D_2 \xrightarrow{d_2} D_3 \ \cdots \ D_{n-1} \xrightarrow{d_{n-1}} D_n$$

of morphisms in $\mathbf{D}$ such that, for each k, either $d_k = \overline{G}^{D_k} a_k$ or $d_k = \overline{F}^{D_k} b_k$, where $a_k = \overline{G} d_k$ and $b_k = \overline{F} d_k$. We will inductively construct the dashed morphisms in the diagram

$$\begin{tikzcd}
D'_1 \arrow[r, dashed, "\overline{F}^{D'_1} b_1"] \arrow[d, equal, "\overline{G}^{D'_1} u_1"'] & D'_2 \arrow[r, dashed, "\overline{F}^{D'_2} b_2"] \arrow[d, dashed, "\overline{G}^{D'_2} u_2"'] & D'_3 \arrow[d, dashed, "\overline{G}^{D'_3} u_3"'] & \cdots & D'_{n-1} \arrow[r, dashed, "\overline{F}^{D'_{n-1}} b_{n-1}"] \arrow[d, dashed, "\overline{G}^{D'_{n-1}} u_{n-1}"] & D'_n \arrow[d, dashed, "\overline{G}^{D'_n} u_n"] \\
D_1 \arrow[r, "d_1"'] & D_2 \arrow[r, "d_2"'] & D_3 & \cdots & D_{n-1} \arrow[r, "d_{n-1}"'] & D_n
\end{tikzcd}$$

in $\mathbf{D}$ such that the resulting diagram in $\mathbf{D}$ commutes and $\overline{F}\,\overline{G}^{D'_k} u_k = \mathrm{id}_{\overline{F}D_k}$ for each k.

For the base step, we may set $D'_1 = D_1$ and $u_1 = \mathrm{id}_{\overline{G}D_1}$, so that $\overline{F}\,\overline{G}^{D'_1} u_1 = \overline{F}\,\mathrm{id}_{D_1} = \mathrm{id}_{\overline{F}D_1}$.

For the inductive step, suppose that we have already constructed D'_k and u_k, and wish now to construct D'_{k+1} and u_{k+1}. Consider the diagram

$$\begin{tikzcd}
& D'_k \arrow[r, dashed, "\overline{F}^{D'_k} b_k"] \arrow[dd, "\overline{G}^{D'_k} u_k"'] & D'_{k+1} \arrow[d, dashed, "\overline{G}^{D'_{k+1}} u_{k+1}"] & & B_k \arrow[r, "b_k"] \arrow[dd, equal] & B_{k+1} \arrow[dd, equal] & \\
\mathbf{D} & & \overline{D_{k+1}} & {} \arrow[r, mapsto, "\overline{F}"] & {} & & \mathbf{B} \\
& D_k \arrow[r, "d_k"'] & D_{k+1} & & B_k \arrow[r, "b_k"'] & B_{k+1} & \\
& {} \arrow[d, mapsto, "\overline{G}"'] & & & {} \arrow[d, mapsto, "G"] & & \\
& {} & & & {} & & \\
& A'_k \arrow[r, dashed, "F^{A'_k} c_k"] \arrow[dd, "u_k"'] & A'_{k+1} \arrow[dd, dashed, "u_{k+1}"] & & C_k \arrow[r, "c_k"] \arrow[dd, equal] & C_{k+1} \arrow[dd, equal] & \\
\mathbf{A} & & & {} \arrow[r, mapsto, "F"'] & {} & & \mathbf{C} \\
& A_k \arrow[r, "a_k"'] & A_{k+1} & & C_k \arrow[r, "c_k"'] & C_{k+1} &
\end{tikzcd}$$

From the universal property of the F-opcartesian morphism $F^{A'_k} c_k$, there is a unique morphism $u_{k+1} \colon A'_{k+1} \to A_{k+1}$ in $\mathbf{A}$ above $\mathrm{id}_{C_{k+1}}$ such that the square in $\mathbf{A}$ above commutes.

Suppose that $d_k = \overline{G}^{D_k} a_k$. By the PutPut axiom, and commutativity of the lens square,

$$d_k \circ \overline{G}^{D'_k} u_k = \overline{G}^{D'_{k+1}} u_{k+1} \circ \overline{G}^{D'_k} F^{A'_k} c_k = \overline{G}^{D'_{k+1}} u_{k+1} \circ \overline{F}^{D'_k} G^{B_k} c_k. \tag{4}$$

We also have $\overline{F}\,\overline{G}^{D'_k} u_k = G^{B_k} F u_k = G^{B_k} \mathrm{id}_{C_k} = \mathrm{id}_{B_k}$ by compatibility of the lens square, and similarly $\overline{F}\,\overline{G}^{D'_{k+1}} u_{k+1} = \mathrm{id}_{B_{k+1}}$. Hence, applying $\overline{F}$ to both sides of (4), we see that $b_k = G^{B_k} c_k$. Thus (4) actually says that $\overline{D_{k+1}} = D_{k+1}$ and the square in $\mathbf{D}$ above commutes.

Otherwise, $d_k = \overline{F}^{D_k} b_k$, and thus also $a_k = \overline{G}\,\overline{F}^{D_k} b_k$. Additionally, as the lens square is compatible, $F^{A'_k} c_k = \overline{G}\,\overline{F}^{D'_k} b_k$. As $(\overline{G}, \overline{F})$ is split independent, it follows again that $\overline{D_{k+1}} = D_{k+1}$ and the square in $\mathbf{D}$ above commutes. $\square$

4.5. Proxy pullbacks of discrete opfibrations and proxy products

As all discrete opfibrations are split opfibrations, the results in the previous section apply in particular to proxy pullbacks of lens cospans with one leg a discrete opfibration. Actually, by better understanding the compatibility and independence of the lens spans that form commuting squares with such a cospan, we may simplify Proposition 4.15 as follows.

Proposition 4.21. *Proxy pullbacks of discrete opfibrations are real pullbacks in $\mathcal{L}ens$.*

Lemma 4.22. *Consider a compatible lens square*

$$\begin{tikzcd} \mathbf{D} \arrow[r, "\overline{F}"] \arrow[d, "\overline{G}"'] & \mathbf{B} \arrow[d, "G"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{C} \end{tikzcd}.$$

If F or G is a discrete opfibration, then $(\overline{G}, \overline{F})$ is independent.

Proof. Without loss of generality, suppose that F is a discrete opfibration. Let $D_1 \in |\mathbf{D}|$, let $a_1 \colon \overline{G}D_1 = A_1 \to A_1'$ in $\mathbf{A}$, let $b \colon \overline{F}D_1 = B_2 \to B_2$ in $\mathbf{B}$, and let $a_2 \colon \operatorname{tgt} \overline{G}\overline{F}^{D_1}b = A_2 \to A_2'$, as shown in the diagram (1). Suppose also that the square in $\mathbf{A}$ commutes, and that $\overline{F}\overline{G}^{D_1}a_1 = \mathrm{id}_{B_1}$ and $\overline{F}\overline{G}^{D_2}a_2 = \mathrm{id}_{B_2}$. We have that

$$Fa_1 = GG^{B_1}Fa_1 = G\overline{F}\overline{G}^{D_1}a_1 = G\,\mathrm{id}_{B_1} = \mathrm{id}_{GB_1} = \mathrm{id}_{FA_1}$$

by compatibility of the lens square, and so $a_1 = \mathrm{id}_{A_1}$ as F is a discrete opfibration. Hence $\overline{G}^{D_1}a_1 = \overline{G}^{D_1}\mathrm{id}_{A_1} = \mathrm{id}_{D_1}$ and $D_1' = D_1$. Similarly, $\overline{G}^{D_2}a_2 = \mathrm{id}_{D_2}$ and $D_2' = D_2$. As $D_1' = D_1$, we have $\overline{F}^{D_1}b = \overline{F}^{D_1'}b$. Hence the square in $\mathbf{D}$ commutes. □

Proof of Proposition 4.21. By Proposition 4.15, a proxy-pullback span of a lens cospan with one leg a discrete opfibration is terminal amongst the independent lens spans that are compatible with the cospan. Every lens span that forms a commuting square with such a cospan is actually compatible with the cospan by Proposition 3.9 and independent by Lemma 4.22. □

Specialising further, we now consider the proxy pullbacks of those cospans whose apex is the terminal category, that is, proxy products. Recall that the unique lens from a category $\mathbf{C}$ to the terminal category is a discrete opfibration if and only if $\mathbf{C}$ is a discrete category. The specialisation of Proposition 4.21 then says that the proxy product of a category with a discrete category is a real product. Actually, in this case, the converse also holds.

Proposition 4.23. *The proxy product of two categories is a real product if and only if at least one of the two categories is a discrete category.*

To prove the converse, it suffices to show, for all non-discrete categories $\mathbf{A}$ and $\mathbf{B}$, that there is a non-independent lens span from $\mathbf{A}$ to $\mathbf{B}$. We may explicitly describe such a non-independent lens span; it is merely the so-called *funny tensor product* $\mathbf{A} \,\square\, \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ with a canonical lens structure on the projection functors. Henceforth, we will refer to the funny tensor product as the *free product* of categories, as it generalises the well-known free product of groups. The free product of categories has several different descriptions; we will use the following one.

Definition 4.24. The *free product* $\mathbf{A} \mathbin{\square} \mathbf{B}$ of categories $\mathbf{A}$ and $\mathbf{B}$ is the category with object set $|\mathbf{A}| \times |\mathbf{B}|$ whose morphisms are freely generated by those of the form

$$(A_1, B) \xrightarrow{(a,B)} (A_2, B) \qquad \text{and} \qquad (A, B_1) \xrightarrow{(A,b)} (A, B_2),$$

subject to the equations

$$\begin{aligned} (\mathrm{id}_A, B) &= \mathrm{id}_{(A,B)} & \qquad (a_2, B) \circ (a_1, B) &= (a_2 \circ a_1, B) \\ (A, \mathrm{id}_B) &= \mathrm{id}_{(A,B)} & \qquad (A, b_2) \circ (A, b_1) &= (A, b_2 \circ b_1). \end{aligned}$$

There are projection lenses $P_1 \colon \mathbf{A} \mathbin{\square} \mathbf{B} \to \mathbf{A}$ and $P_2 \colon \mathbf{A} \mathbin{\square} \mathbf{B} \to \mathbf{B}$, defined by the equations

$$\begin{array}{llll} P_1(A, B) = A & P_1(a, B) = a & P_1(A, b) = \mathrm{id}_A & P_1{}^{(A,B)}a = (a, B) \\ P_2(A, B) = B & P_2(A, b) = b & P_2(a, B) = \mathrm{id}_B & P_2{}^{(A,B)}b = (A, b), \end{array}$$

whose get functors are the usual projection functors.

Proof of Proposition 4.23. The *if* direction is a particular case of Proposition 4.21. For the *only if* direction, suppose that $\mathbf{A}$ and $\mathbf{B}$ are both non-discrete categories, that is, that there are non-identity morphisms $a \colon A_1 \to A_2$ in $\mathbf{A}$ and $b \colon B_1 \to B_2$ in $\mathbf{B}$. Then the morphisms

$$(A_1, B_1) \xrightarrow{(a,B_1)} (A_2, B_1) \xrightarrow{(A_2,b)} (A_2, B_2) \qquad \text{and} \qquad (A_1, B_1) \xrightarrow{(A_1,b)} (A_1, B_2) \xrightarrow{(a,B_2)} (A_2, B_2)$$

in $\mathbf{A} \mathbin{\square} \mathbf{B}$ both have the same source object, and both are mapped by P_1 to a and P_2 to b, but they are not equal. Hence the lens span $\mathbf{A} \xleftarrow{P_1} \mathbf{A} \mathbin{\square} \mathbf{B} \xrightarrow{P_2} \mathbf{B}$ is not independent. $\square$

We conclude this chapter with a few remarks about the connections between the results in this chapter and other works. Proposition 4.21 and the *if* direction of Proposition 4.23 are already known to Chollet et al. [8]. Proposition 4.10, which is arguably the central result of this chapter, is very closely related to Böhm's relative pullback [6] and Simpson's local independent product [29]. To view a sync-minimal proxy pullback of a lens cospan as a pullback relative to the class of independent spans that are compatible with the cospan, we would need to generalise the notion of relative pullback to work relative to chosen classes of spans on each cospan rather than merely relative to a single overall class of spans. To view a sync-minimal proxy pullback of a cospan as a local independent product, we would need lens-multispan generalisations of the lens-span notions of independence and compatibility in order to define the appropriate *local independence structure* on $\mathcal{L}ens$. Working out the details and implications of these connections to the work of Böhm and Simpson is left as future work.

CHAPTER 5

Monic and epic lenses

In this chapter, we will give complete and elementary characterisations of the monos and epis in $\mathcal{L}ens$, confirming several conjectures by Chollet et al. [8]. These will allow us to deduce

- that $\mathcal{L}ens$ has an (epi, mono) factorisation system,
- that monos, epis and image factorisations in $\mathcal{L}ens$ are proxy-pullback stable, and
- that a proxy analogue of the kernel-pair characterisation of monos holds.

The last two of the three results listed above are further examples of ways in which the proxy pullback in $\mathcal{L}ens$ behaves like a real pullback. We will also make extensive use of the epi characterisation when we study the coequalisers in $\mathcal{L}ens$ in Chapter 6. In particular, it is essential to our proof of Theorem 6.6, which is arguably the main result of that chapter.

5.1. Monic lenses

We will study the monos in $\mathcal{L}ens$ via their relation to those in $\mathcal{C}at$, expressed as follows.

THEOREM 5.1. *The functor $\mathcal{G}$ preserves and reflects monos.*

Reflection was proved and preservation conjectured by Chollet et al. [8]. For the proof, recall that a morphism is monic if and only if it has a kernel pair with both morphisms equal.

PROOF THAT $\mathcal{G}$ PRESERVES MONOS. Let M be a monic lens, and let (P_1, P_2) be its proxy kernel pair. As M is monic and $M \circ P_1 = M \circ P_2$, actually $P_1 = P_2$, and so $\mathcal{G}P_1 = \mathcal{G}P_2$. But $(\mathcal{G}P_1, \mathcal{G}P_2)$ is the (real) kernel pair of $\mathcal{G}M$ in $\mathcal{C}at$. Hence $\mathcal{G}M$ is a monic functor. $\square$

Definition 5.2. A *cosieve* is an injective-on-objects discrete opfibration.

Chollet et al. [8] also showed that the get functor of a lens is monic if and only if the lens is a cosieve. The following corollary to Theorem 5.1 is an extension of Chollet et al.'s result; it says that monic lenses and cosieves are essentially the same. We will continue to use the term cosieve for functors when we wish to distinguish these from monic lenses.

Corollary 5.3. *The functor $\mathcal{G}$ restricts to a bijection between monic lenses and cosieves.*

PROOF. Let F be a monic lens. Then $\mathcal{G}F$ is a monic functor as $\mathcal{G}$ preserves monos. Monic functors are injective on both objects and morphisms. Also, all injective-on-morphisms lenses are discrete opfibrations. Hence $\mathcal{G}F$ is a cosieve.

Let $\overline{F}$ be a cosieve. As $\overline{F}$ is a discrete opfibration, there is a unique lens F such that $\mathcal{G}F = \overline{F}$. As the functor $\overline{F}$ is monic and $\mathcal{G}$ reflects monos, the lens F is also monic. $\square$

Remark 5.4. As monic lenses are discrete opfibrations, by Proposition 4.21, proxy pullbacks along monos are real pullbacks. This also means that monos are proxy-pullback stable.

Knowing now that $\mathcal{G}$ preserves and reflects monos, we may show that the kernel-pair characterisation of monos lifts from $\mathcal{C}at$ to an analogous proxy-kernel-pair characterisation of the monos in $\mathcal{L}ens$. This is yet another pullback-like property of the proxy pullback.

Proposition 5.5. *Let F be a lens, and let (P_1, P_2) be a proxy kernel pair of F. Then F is monic if and only if $P_1 = P_2$, in which case P_1 is an isomorphism.*

Proof. Suppose that $P_1 = P_2$. As $(\mathcal{G}P_1, \mathcal{G}P_2)$ is a kernel pair of $\mathcal{G}F$ and $\mathcal{G}P_1 = \mathcal{G}P_2$, the functor $\mathcal{G}F$ is monic. As $\mathcal{G}$ reflects monos, the lens F is also monic.

Conversely, suppose that F is monic. Then $P_1 = P_2$ because $F \circ P_1 = F \circ P_2$. Also, as $\mathcal{G}$ preserves monos, the functor $\mathcal{G}F$ is monic, and so the functor $\mathcal{G}P_1$ is an isomorphism. As $\mathcal{G}$ is conservative, the lens P_1 is also an isomorphism. □

Corollary 5.6. *A lens $F\colon \mathbf{A} \to \mathbf{B}$ is monic if and only if $(\mathrm{id}_{\mathbf{A}}, \mathrm{id}_{\mathbf{A}})$ is a proxy kernel pair of F.*

Proof. We begin with the *only if* direction. Suppose that F is monic. Recall that every lens has a proxy kernel pair. By Proposition 5.5, F has one of the form (P, P) where P is an isomorphism. The lens P is a lens span isomorphism from (P, P) to $(\mathrm{id}_{\mathbf{A}}, \mathrm{id}_{\mathbf{A}})$, so $(\mathrm{id}_{\mathbf{A}}, \mathrm{id}_{\mathbf{A}})$ is another proxy kernel pair of F. The *if* direction follows immediately from Proposition 5.5. □

5.2. Lens images and factorisation

The images of the object and morphism maps of a functor do not always form a subcategory of the target category of the functor. The get functor of a lens F is better behaved; in this case, the images actually form an out-degree-zero subcategory $\mathbf{Im}\, F$ of the target category of the lens, which we will call the *image* of F. By *out-degree-zero* subcategory, we mean one for which any morphism out of an object in the subcategory belongs to the subcategory. As cosieves are exactly the out-degree-zero subcategory inclusion functors, we obtain the following result.

Proposition 5.7. *Every lens $F\colon \mathbf{A} \to \mathbf{B}$ has a factorisation*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "E"'] \arrow[rr, bend left, "F"] & \mathbf{Im}\, F \arrow[r, "M"'] & \mathbf{B} \end{tikzcd}$$

in $\mathcal{L}ens$ where M is monic and E is surjective on objects and morphisms.

Recall that a morphism $e\colon A \to B$ is *left orthogonal* to a morphism $m\colon C \to D$, written $e \perp m$, if, for all pairs of morphisms $f\colon A \to C$ and $g\colon B \to D$ such that $g \circ e = m \circ f$, there is a unique morphism $h\colon B \to C$, called the *diagonal filler*, such that $f = h \circ e$ and $g = m \circ h$.

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "h"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

Also recall that a pair $(\mathscr{E}, \mathscr{M})$ of classes of morphisms is a *(orthogonal) factorisation system* if

- every morphism f factors as $f = m \circ e$ for some $e \in \mathscr{E}$ and some $m \in \mathscr{M}$,
- $\mathscr{E}$ is the class of morphisms e such that $e \perp m$ for all $m \in \mathscr{M}$, and
- $\mathscr{M}$ is the class of morphisms m such that $e \perp m$ for all $e \in \mathscr{E}$.

Remark 5.8. The factorisation in Proposition 5.7 is already known to Johnson and Rosebrugh; they observed that the surjective-on-objects lenses and the injective-on-objects-and-morphisms lenses form a factorisation system on $\mathcal{L}ens$ [18]. We claim that it is actually an (epi, mono) factorisation system—we have already shown that the injective-on-objects-and-morphisms lenses are exactly the monic lenses and we will show in the next section that the surjective-on-objects lenses are exactly the epic lenses. In Section 6.4, we will deduce the orthogonality conditions without explicitly constructing diagonal fillers. The orthogonality conditions imply that the factorisation in Proposition 5.7 is in fact an image factorisation.

5.3. Epic lenses

We may also study the epis in $\mathcal{L}ens$ via their relation to those in $\mathcal{C}at$.

THEOREM 5.9. *The functor $\mathcal{G}$ preserves and reflects epis.*

Again, reflection was proved and preservation conjectured by Chollet et al. [8]. Lack sketched a proof of preservation in an unpublished personal communication to Clarke; we present a new, simpler proof below. First, we recall some preliminary results about epic functors and epic lenses.

Proposition 5.10. *Every epic functor is surjective on objects. Every functor that is surjective both on objects and on morphisms is epic.*

Recall that not all epic functors are surjective on morphisms.

Example 5.11. Let $J \colon \mathbf{2} \to \mathbf{I}$ be the functor that sends the non-identity morphism u of the interval category $\mathbf{2}$ to the morphism v of the free living isomorphism $\mathbf{I}$. Then J is epic because any two functors out of $\mathbf{I}$ which agree on v must also agree on v^{-1}. However, the morphism v^{-1} is not in the image of J.

Proposition 5.12. *Let F be a lens, and let $\overline{J}_1, \overline{J}_2$ be the cokernel pair of $\mathcal{G}F$. Then $\overline{J}_1$ and $\overline{J}_2$ are cosieves, and the unique lenses J_1 and J_2 above $\overline{J}_1$ and $\overline{J}_2$ satisfy $J_1 \circ F = J_2 \circ F$.*

PROOF. Let $F = M \circ E$ be the factorisation of F given in Proposition 5.7. By Proposition 5.10, $\mathcal{G}E$ is an epic functor. As $\overline{J}_1 \circ \mathcal{G}M \circ \mathcal{G}E = \overline{J}_1 \circ \mathcal{G}F = \overline{J}_2 \circ \mathcal{G}F = \overline{J}_2 \circ \mathcal{G}M \circ \mathcal{G}E$, actually $\overline{J}_1 \circ \mathcal{G}M = \overline{J}_2 \circ \mathcal{G}M$. It follows that $\mathcal{G}M$ also has cokernel pair $\overline{J}_1, \overline{J}_2$. As cosieves are pushout stable and $\mathcal{G}M$ is a cosieve, so are $\overline{J}_1$ and $\overline{J}_2$. As there is a unique lens above the discrete opfibration $\overline{J}_1 \circ \mathcal{G}M = \overline{J}_2 \circ \mathcal{G}M$, we must have that $J_1 \circ M = J_2 \circ M$. □

Remark 5.13. Later, we will see that J_1 and J_2 are actually a cokernel pair of F in $\mathcal{L}ens$.

PROOF THAT $\mathcal{G}$ PRESERVES EPIS. Let E be an epic lens, and J_1 and J_2 the unique lenses above the cokernel pair of $\mathcal{G}E$ from Proposition 5.12. As $J_1 \circ E = J_2 \circ E$ and E is epic, actually $J_1 = J_2$, and so $\mathcal{G}J_1 = \mathcal{G}J_2$. But $\mathcal{G}J_1$ and $\mathcal{G}J_2$ are the cokernel pair of $\mathcal{G}E$, so $\mathcal{G}E$ is also epic. □

Corollary 5.14. *Let F be a lens. Then the following are equivalent:*

(1) *F is epic,*
(2) *$\mathcal{G}F$ is surjective on objects,*
(3) *$\mathcal{G}F$ is surjective on morphisms.*

PROOF. Chollet et al. [8, Proposition 4.15] showed that (2) and (3) are equivalent, and imply (1). Suppose that F is epic. As $\mathcal{G}$ preserves epis (Theorem 5.9), the functor $\mathcal{G}F$ is also epic. By Proposition 5.10, $\mathcal{G}F$ is surjective on objects. □

Proposition 5.15. *Epic lenses are proxy pullback stable.*

PROOF. Consider the proxy-pullback square in $\mathcal{L}ens$ depicted below.

$$\begin{tikzcd} \mathbf{D} \arrow[r, "\overline{F}"] \arrow[d, "\overline{E}"'] \arrow[dr, phantom, "\mathsf{PPB}"] & \mathbf{B} \arrow[d, "E"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{C} \end{tikzcd}$$

Suppose that E is epic. Let $A \in |\mathbf{A}|$. As E is surjective on objects, there is a $B \in |\mathbf{B}|$ such that $EB = FA$. As the square of get functors is a pullback square in $\mathcal{C}at$, there is a unique $D \in |\mathbf{D}|$ such that $\overline{E}D = A$ and $\overline{F}D = B$. Hence $\overline{E}$ is surjective on objects, and thus epic. □

Remark 5.16. As image factorisations in $\mathcal{L}ens$ come from the (epi, mono) factorisation system and the epis and monos in $\mathcal{L}ens$ are proxy-pullback stable, so are the image factorisations.

CHAPTER 6

Coequalisers of lenses

In this chapter, we study the coequalisers in $\mathcal{L}ens$. We will see in Section 6.1 that they are, in general, not so well behaved—not all parallel pairs of lenses have coequalisers and the forgetful functor from $\mathcal{L}ens$ to $\mathcal{C}at$ neither preserves nor reflects them. We will, however, obtain two surprising results about classes of coequalisers in $\mathcal{L}ens$ that *do* exist and lie over coequalisers in $\mathcal{C}at$. The first is Theorem 6.12, which says that $\mathcal{L}ens$ has pushouts of monic lenses with discrete opfibrations. The second is Corollary 6.21, which says that every epic lens is *proxy effective*, that is, coequalises its proxy kernel pair. Our proofs of Theorem 6.12 and Corollary 6.21 both depend on Theorem 6.6—a general result about the coequalisers that are actually reflected by $\mathcal{G}$. Important corollaries of Theorem 6.12 and Corollary 6.21 include

- that every monic lens is effective;
- that the classes of all monos, all effective monos, all regular monos, all strong monos and all extremal monos in $\mathcal{L}ens$ coincide;
- that the classes of all epis, all proxy-effective epis, all regular epis, all strong epis and all extremal epis in $\mathcal{L}ens$ coincide;
- that all lenses that are both monic and epic are isomorphisms; and
- that the orthogonality conditions for the (epi, mono) factorisation system on $\mathcal{L}ens$ hold.

Remark 6.1. We say that a morphism $e\colon B \to C$ *coforks* a pair of morphisms $f_1, f_2\colon A \to B$ if $e \circ f_1 = e \circ f_2$. Some authors would use the verb coequalise where we use the verb cofork. Unlike those authors, we say that e *coequalises* f_1 and f_2 only when e is a universal cofork of f_1 and f_2.

6.1. Non-existence, non-preservation and non-reflection of coequalisers

Recall that $\mathcal{C}at$ has all coequalisers. Shortly, we will see examples of how coequalisers in $\mathcal{L}ens$ are not so well behaved. The following proposition, which gives necessary conditions for a cofork of lenses to be a coequaliser, will be helpful.

Proposition 6.2. *Let $F_1, F_2\colon \mathbf{A} \to \mathbf{B}$ be lenses with coequaliser $E\colon \mathbf{B} \to \mathbf{C}$ in $\mathcal{L}ens$. Then*

(1) *for each cofork $G\colon \mathbf{B} \to \mathbf{D}$ of F_1 and F_2, $G^B d = E^B E G^B d$ for all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$; and*

(2) *in particular, E is the unique lens above $\mathcal{G}E$ that coforks F_1 and F_2.*

Proof. For (1), if $G\colon \mathbf{B} \to \mathbf{D}$ coforks F_1 and F_2, then there is a lens $H\colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$, and so $G^B d = E^B H^{EB} d = E^B E E^B H^{EB} d = E^B E G^B d$. For (2), if $G\colon \mathbf{B} \to \mathbf{C}$ is a lens above $\mathcal{G}E$ that coforks F_1 and F_2, then, for each $B \in |\mathbf{B}|$ and each $c \in \mathbf{C}(EB, *)$, we have that $G^B c = E^B E G^B c = E^B G G^B c = E^B c$, and so $G = E$. $\square$

The first example shows that $\mathcal{L}ens$ does not have all coequalisers, nor does $\mathcal{G}$ reflect them.

Example 6.3. Let $\mathbf{A}$ and $\mathbf{B}$ be the categories generated respectively by the following graphs.

$$\begin{tikzcd} Y_1 & X \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "f"] & Y_2 \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} Y_1' & X' \arrow[l, "f_1'"'] \arrow[r, "f_2'"] & Y_2' \end{tikzcd}$$

Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be the unique lenses that both send X to X', Y_1 to Y_1', Y_2 to Y_2', and such that $F_1 Y = Y_1'$, ${F_1}^X f_1' = f_1$, $F_2 Y = Y_2'$, and ${F_2}^X f_2' = f_2$. Let $G \colon \mathbf{B} \to \mathbf{2}$ be the unique functor that sends X' to 0, and both Y_1' and Y_2' to 1; G coequalises $\mathcal{G}F_1$ and $\mathcal{G}F_2$ in $\mathcal{C}at$. There are only two lens structures on G that cofork F_1 and F_2 in $\mathcal{L}ens$; one is determined by ${G_1}^{X'} u = f_1'$ and the other by ${G_2}^{X'} u = f_2'$. By Proposition 6.2, neither G_1 nor G_2 coequalises F_1 and F_2. Thus $\mathcal{G}$ does not reflect the coequaliser G of $\mathcal{G}F_1$ and $\mathcal{G}F_2$.

Actually F_1 and F_2 do not have a coequaliser in $\mathcal{L}ens$. Assume that $E \colon \mathbf{B} \to \mathbf{C}$ is such a coequaliser. Then $Ef_1' = EF_1 f = EF_2 f = Ef_2'$. As G_1 coforks F_1 and F_2, there is a lens $H \colon \mathbf{C} \to \mathbf{2}$ such that $G_1 = H \circ E$. As $HEX' = G_1 X' \neq G_1 Y_1' = HEY_1'$, we must have $EX' \neq EY_1'$. Hence EX' and EY_1' are distinct objects of the image of E, and $\mathrm{id}_{EX'}$, Ef_1' and $\mathrm{id}_{EY_1'}$ are distinct morphisms of the image of E. As E is a coequaliser, it is epi, and so, by Corollary 5.14, its image is all of $\mathbf{C}$. Thus $\mathcal{G}H$ is an isomorphism in $\mathcal{C}at$, and so H is an isomorphism in $\mathcal{L}ens$. Hence G_1 also coequalises F_1 and F_2, which is a contradiction.

There are even parallel pairs of lenses for which the coequaliser of their get functors has a unique lens structure that coforks them, and yet does not coequalise them.

Example 6.4. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be the preorded sets generated by the graphs

$$\begin{tikzcd} Z_1 & X \arrow[l, "h_1"'] \arrow[r, "f"] \arrow[d, "h"'] \arrow[rr, bend right, "h_2"'] & Y \arrow[r, "g"] & Z_2 \\ & Z & & \end{tikzcd} \qquad \begin{tikzcd} Z_1' & X' \arrow[l, "h_1'"'] \arrow[r, "f'"] \arrow[rr, bend right, "h_2'"'] & Y' \arrow[r, "g'"] & Z_2' \end{tikzcd} \qquad \begin{tikzcd} X'' \arrow[r, "f''"] \arrow[rr, bend right, "h''"'] & Y'' \arrow[r, "g''"] & Z'' \end{tikzcd}$$

respectively. Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be the unique lenses that both send X to X', Y to Y', Z_1 to Z_1', Z_2 to Z_2', and such that $F_1 Z = Z_1'$, ${F_1}^X h_1' = h_1$ and $F_2 Z = Z_2'$. Let $E \colon \mathbf{B} \to \mathbf{C}$ be the unique lens that sends X' to X'', Y' to Y'', and both Z_1' and Z_2' to Z''. Then $\mathcal{G}E$ coequalises $\mathcal{G}F_1$ and $\mathcal{G}F_2$ in $\mathcal{C}at$, and E coforks F_1 and F_2 in $\mathcal{L}ens$. However, E does not coequalise F_1 and F_2 in $\mathcal{L}ens$. Indeed, if $G \colon \mathbf{B} \to \mathbf{2}$ is the unique lens that sends X' to 0, all of Y', Z_1' and Z_2' to 1, and for which $G^{X'} u = h_1'$, then $E^{X'} E G^{X'} u = E^{X'} E h_1' = E^{X'} h'' = h_2' \neq h_1' = G^{X'} u$.

The final example shows that $\mathcal{G}$ does not preserve coequalisers. It is also an example of a parallel pair of lenses whose get functors' coequaliser has no lens structure that coforks them.

Example 6.5. Let $\mathbf{A}$ be the preordered set generated by the graph

$$\begin{tikzcd} Y_1 & X \arrow[l, "f_1"'] \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

Let $I \colon \mathbf{A} \to \mathbf{A}$ denote the identity lens, and let $S \colon \mathbf{A} \to \mathbf{A}$ be the unique lens that maps X to X, Y_1 to Y_2 and Y_2 to Y_1. The coequaliser of $\mathcal{G}I$ and $\mathcal{G}S$ in $\mathcal{C}at$ is the unique functor $\mathbf{A} \to \mathbf{2}$

that sends X to 0 and both Y_1 and Y_2 to 1. Recall that $\mathbf{1}$ is terminal in $\mathcal{L}ens$ [8]. We claim that the coequaliser of I and S in $\mathcal{L}ens$ is the unique lens $E\colon \mathbf{A} \to \mathbf{1}$. Let $G\colon \mathbf{A} \to \mathbf{C}$ be a lens that coforks I and S in $\mathcal{L}ens$. Let $f = Gf_1$. Then $f = Gf_1 = GIf_1 = GSf_1 = Gf_2$. As $G^X f \in \mathbf{A}(X, *)$, it is one of f_1, f_2 and id_X. If $G^X f = f_1$, then

$$f_1 = I^X f_1 = I^X G^X f = (G \circ I)^X f = (G \circ S)^X f = S^X G^X f = S^X f_1 = f_2,$$

which is a contradiction. We get a similar contradiction if $G^X f = f_2$. By elimination, $G^X f = \mathrm{id}_X$, and so $f = GG^X f = G\,\mathrm{id}_X = \mathrm{id}_{GX}$. The image of G thus consists of the object GX and the morphism id_{GX}. If $H\colon \mathbf{1} \to \mathbf{C}$ is a lens such that $G = H \circ E$, then H must send 0 to GX, and this uniquely determines H. As the image of any lens, in particular G, is an out-degree-zero subcategory of its target category, this definition of H does indeed give a lens, and $G = H \circ E$. Of course, the factorisation $G = H \circ E$ is really the image factorisation of G from Remark 5.8.

6.2. Coequalisers which are reflected

Although the counterexamples above suggest that coequalisers in $\mathcal{L}ens$ have little relation to those in $\mathcal{C}at$, we will see in Theorem 6.12 and Corollary 6.21 two classes of coequalisers in $\mathcal{L}ens$ which do lie over coequalisers in $\mathcal{C}at$. The following theorem, a partial converse to Proposition 6.2, reduces checking the coequaliser property in these cases to checking that (1) below always holds.

THEOREM 6.6. *Let $E\colon \mathbf{B} \to \mathbf{C}$ cofork $F_1, F_2\colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{L}ens$. Suppose that $\mathcal{G}E$ coequalises $\mathcal{G}F_1$ and $\mathcal{G}F_2$ in $\mathcal{C}at$. Then E coequalises F_1 and F_2 in $\mathcal{L}ens$ if and only if, for all lenses $G\colon \mathbf{B} \to \mathbf{D}$ that cofork F_1 and F_2 in $\mathcal{L}ens$, all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$, we have*

$$G^B d = E^B E G^B d. \tag{1}$$

In the proof of the following lemma and again, later, in the proof of Lemma 6.11, we use the induction principle for the equivalence relation $\simeq$ on a set S generated by a binary relation R on S. Explictly, the induction principle is given by the following logical formula.

$$\forall P\ x_0\ y_0.\ \left[\begin{array}{l} x_0 \simeq y_0 \\ \quad \wedge \quad \forall x\ y.\ x\ R\ y \implies P(x,y) \\ \quad \wedge \quad \forall x.\ P(x,x) \\ \quad \wedge \quad \forall x\ y.\ [x \simeq y\ \wedge\ P(x,y)] \implies P(y,x) \\ \quad \wedge \quad \forall x\ y\ z.\ [x \simeq y\ \wedge\ P(x,y)\ \wedge\ y \simeq z\ \wedge\ P(y,z)] \implies P(x,z) \end{array}\right] \implies P(x_0, y_0) \tag{2}$$

Lemma 6.7. *Let $F_1, F_2\colon \mathbf{A} \to \mathbf{B}$ be lenses. Let $E\colon \mathbf{B} \to \mathbf{C}$ be a cofork of F_1 and F_2 in $\mathcal{L}ens$, and suppose that $\mathcal{G}E$ coequalises $\mathcal{G}F_1$ and $\mathcal{G}F_2$ in $\mathcal{C}at$. Let $G\colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2 in $\mathcal{L}ens$, and let $H\colon \mathbf{C} \to \mathbf{D}$ be the unique functor such that $\mathcal{G}G = H \circ \mathcal{G}E$. Then there is a unique lens structure on H such that, for all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$, we have*

$$H^{EB} d = E G^B d. \tag{3}$$

Proof. For each $C \in |\mathbf{C}|$, as $\mathcal{G}E$ is epic, there is a $B \in |\mathbf{B}|$ such that $EB = C$. Hence, we may define H^C using (3), so long as, for all $B_1, B_2 \in |\mathbf{B}|$, if $EB_1 = EB_2$ then, for all $d \in \mathbf{D}(EB_1, *)$, we have $EG^{B_1}d = EG^{B_2}d$. Let $\simeq$ be the smallest equivalence relation on $|\mathbf{B}|$ such that $F_1A \simeq F_2A$ for all $A \in |\mathbf{A}|$. As $\mathcal{G}E$ coequalises $\mathcal{G}F_1$ and $\mathcal{G}F_2$ in $\mathcal{C}at$, we have [5, Proposition 4.1], for all $B_1, B_2 \in |\mathbf{B}|$, that $EB_1 = EB_2$ if and only if $B_1 \simeq B_2$. We proceed using the induction principle in (2). The proof obligations from the reflexivity, symmetry and transitivity axioms for $\simeq$ hold as $=$ is an equivalence relation. For the remaining one, for all $A \in |\mathbf{A}|$ and all $d \in \mathbf{D}(F_1A, *)$, we have

$$EG^{F_1A}d = EF_1{F_1}^AG^{F_1A}d = (E \circ F_1)(G \circ F_1)^Ad$$
$$= (E \circ F_2)(G \circ F_2)^Ad = EF_2{F_2}^AG^{F_2A}d = EG^{F_2A}d.$$

Define H^C using (3). It remains to check the lens axioms. For all $C \in |\mathbf{C}|$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $H^C \operatorname{id}_{HC} = EG^B \operatorname{id}_{GB} = E \operatorname{id}_B = \operatorname{id}_C$; so PutId holds. For all $C \in |\mathbf{C}|$, all $d \in \mathbf{D}(HC, *)$ and all $d' \in \mathbf{D}(\operatorname{tgt} d, *)$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and

$$H^C(d' \circ d) = EG^B(d' \circ d) = E\big(G^{B'}d' \circ G^Bd\big) = EG^{B'}d' \circ EG^Bd = H^{C'}d' \circ H^Cd,$$

where $B' = \operatorname{tgt} G^Bd$ and $C' = EB'$; so PutPut holds. For all $C \in |\mathbf{C}|$ and all $d \in \mathbf{D}(HC, *)$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $HH^Cd = HEG^Bd = GG^Bd = d$; so PutGet holds. ☐

Proof of Theorem 6.6. We proved the *only if* direction in Proposition 6.2. For the *if* direction, suppose, for all lenses $G \colon \mathbf{B} \to \mathbf{D}$ that cofork F_1 and F_2, that (1) always holds. We must show that E is the universal cofork of F_1 and F_2 in $\mathcal{L}ens$. Let $G \colon \mathbf{B} \to \mathbf{D}$ be another cofork of F_1 and F_2 in $\mathcal{L}ens$. Suppose that there is a lens $H \colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$. Then $\mathcal{G}G = \mathcal{G}H \circ \mathcal{G}E$, and so $\mathcal{G}H$ is the unique functor that composes with $\mathcal{G}E$ to give $\mathcal{G}G$. Let $C \in |\mathbf{C}|$ and $d \in \mathbf{D}(HC, *)$. As $\mathcal{G}E$ is epic, there is a $B \in |\mathbf{B}|$ such that $EB = C$. Then $H^Cd = EE^BH^Cd = E(H \circ E)^Bd = EG^Bd$. Hence H is uniquely determined. Now let $H \colon \mathbf{C} \to \mathbf{D}$ be the lens defined as in Lemma 6.7. For all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$, we have $G^Bd = E^BEG^Bd = E^BH^{EB}d = (H \circ E)^Bd$, and so $G = H \circ E$. ☐

Corollary 6.8. *Let $E \colon \mathbf{B} \to \mathbf{C}$ cofork $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{L}ens$. Suppose that $\mathcal{G}E$ coequalises $\mathcal{G}F_1$ and $\mathcal{G}F_2$ in $\mathcal{C}at$. If E is a discrete opfibration then it coequalises F_1 and F_2 in $\mathcal{L}ens$.*

Proof. Apply Theorem 6.6 and the GetPut axiom of the discrete opfibration E. ☐

6.3. Pushouts of discrete opfibrations along monos

In the proof that $\mathcal{G}$ preserves epis (Theorem 5.9), we used the well-known result that cosieves are pushout stable to explain why the pushout in $\mathcal{C}at$ of the get functors of a span of monic lenses lifts uniquely to a commutative square in $\mathcal{L}ens$; this lifted square is actually a pushout square in $\mathcal{L}ens$. In this section, we will show, more generally, that $\mathcal{L}ens$ has pushouts of discrete opfibrations along monics, and that $\mathcal{G}$ creates these pushouts. In what follows, we use square brackets for equivalence classes of elements.

Fritsch and Latch [15, Proposition 5.2] explicitly construct the pushout in $\mathcal{C}at$ of a functor along a full monic functor. We obtain the following simplification of this construction by specialising to pushouts along cosieves, and recalling that the image of a cosieve is out-degree-zero.

Proposition 6.9. *Let $F\colon \mathbf{A} \to \mathbf{C}$ be a functor and $J\colon \mathbf{A} \to \mathbf{B}$ be a cosieve. Then*

$$\begin{tikzcd} \mathbf{A} \arrow[r, tail, "J"] \arrow[d, "F"'] & \mathbf{B} \arrow[d, "\overline{F}"] \\ \mathbf{C} \arrow[r, tail, "\overline{J}"'] & \mathbf{D} \end{tikzcd}$$

is a pushout square in $\mathcal{C}at$ and $\overline{J}$ is a cosieve, where $\mathbf{D}$, $\overline{F}$ and $\overline{J}$ are defined as follows:

- Object set:
$$|\mathbf{D}| = |\mathbf{C}| \sqcup \big(|\mathbf{B}| \setminus |\mathbf{A}|\big)$$
- Hom-sets: *for all $C_1, C_2 \in |\mathbf{C}|$ and all $B_1, B_2 \in |\mathbf{B}| \setminus |\mathbf{A}|$,*
$$\mathbf{D}(C_1, C_2) = \mathbf{C}(C_1, C_2) \qquad \mathbf{D}(C_1, B_2) = \emptyset$$
$$\mathbf{D}(B_1, B_2) = \mathbf{B}(B_1, B_2) \qquad \mathbf{D}(B_1, C_2) = \Big(\coprod_{A \in |\mathbf{A}|} \mathbf{C}(FA, C_2) \times \mathbf{B}(B_1, A)\Big)\Big/\!\sim$$
where $\sim$ is the equivalence relation on $\coprod_{A\in|\mathbf{A}|} \mathbf{C}(FA, C_2) \times \mathbf{B}(B_1, A)$ generated by
$$(c, a \circ b) \sim (c \circ Fa, b)$$
for all $A_1, A_2 \in |\mathbf{A}|$, all $b \in \mathbf{B}(B_1, A_1)$, all $a \in \mathbf{A}(A_1, A_2)$ and all $c \in \mathbf{C}(FA_2, C_2)$.
- Composition: *for all $B_1, B_2, B_3 \in |\mathbf{B}| \setminus |\mathbf{A}|$, all $A \in |\mathbf{A}|$, all $C_1, C_2, C_3 \in |\mathbf{C}|$, all $b_1 \in \mathbf{D}(B_1, B_2)$, all $b_2 \in \mathbf{D}(B_2, B_3)$, all $a \in \mathbf{D}(B_2, A)$, all $c \in \mathbf{D}(FA, C_2)$, all $c_1 \in \mathbf{D}(C_1, C_2)$ and all $c_2 \in \mathbf{D}(C_2, C_3)$,*
$$b_2 \circ_{\mathbf{D}} b_1 = b_2 \circ_{\mathbf{B}} b_1 \qquad [(c, a)] \circ_{\mathbf{D}} b_1 = [(c, a \circ_{\mathbf{B}} b_1)]$$
$$c_2 \circ_{\mathbf{D}} c_1 = c_2 \circ_{\mathbf{C}} c_1 \qquad c_2 \circ_{\mathbf{D}} [(c, a)] = [(c_2 \circ_{\mathbf{C}} c, a)]$$
- Identity morphisms: *same as in $\mathbf{B}$ and $\mathbf{C}$.*
- Injections: *the functor $\overline{J}\colon \mathbf{C} \to \mathbf{D}$ is the obvious inclusion of $\mathbf{C}$ as a full subcategory of $\mathbf{D}$; the functor $\overline{F}\colon \mathbf{B} \to \mathbf{D}$ is defined, for all $B, B' \in |\mathbf{B}| \setminus |\mathbf{A}|$, all $A, A' \in |\mathbf{A}|$, all $b \in \mathbf{B}(B, B')$, all $b' \in \mathbf{B}(B, A)$ and all $a \in \mathbf{B}(A, A')$, as follows:*
$$\overline{F}B = B \qquad \qquad \overline{F}A = FA$$
$$\overline{F}b = b \qquad \overline{F}b' = [(\mathrm{id}_{FA}, b')] \qquad \overline{F}a = Fa$$

Theorem 6.10. *In $\mathcal{C}at$, discrete opfibrations are stable under pushout along cosieves.*

Lemma 6.11. *Let $F\colon \mathbf{A} \to \mathbf{C}$ be a discrete opfibration, let $J\colon \mathbf{A} \to \mathbf{B}$ be a cosieve, let $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and let $C \in |\mathbf{C}|$. Then, for all $A_1, A_2 \in \mathbf{A}$, all $b_1 \in \mathbf{B}(B, A_1)$, all $b_2 \in \mathbf{B}(B, A_2)$, all $c_1 \in \mathbf{C}(FA_1, C)$ and all $c_2 \in \mathbf{C}(FA_2, C)$, if $(c_1, b_1) \sim (c_2, b_2)$ then $F^{A_1}c_1 \circ b_1 = F^{A_2}c_2 \circ b_2$.*

Proof. We proceed by induction, using the induction principle for $\sim$ in (2). The proof obligations from the reflexivity, symmetry and transitivity axioms for $\sim$ hold because $=$ is an equivalence relation. For the remaining proof obligation, for all $A_1, A_2 \in |\mathbf{A}|$, all $b \in \mathbf{B}(B, A_1)$, all $a \in \mathbf{A}(A_1, A_2)$ and all $c \in \mathbf{C}(FA_2, C)$, we have $F^{A_1}Fa = a$ as F is a discrete opfibration, and so $F^{A_2}c \circ (a \circ b) = F^{A_2}c \circ F^{A_1}Fa \circ b = F^{A_1}(c \circ Fa) \circ b$. $\square$

Proof of Theorem 6.10. With the notation of Proposition 6.9, suppose that F is a discrete opfibration. We must show that $\overline{F}$ is a discrete opfibration. Let $B \in |\mathbf{B}|$ and $d \in \mathbf{D}(\overline{F}B, *)$.

Suppose that $B \in |\mathbf{A}|$. Then $\overline{F}B = FB$, and $d \in \mathbf{C}(FB, *)$. As F is a discrete opfibration, there is a unique $a \in \mathbf{A}(B, *)$ with $d = Fa$. But $\mathbf{A}(B, *) = \mathbf{B}(B, *)$ as $\mathbf{A}$ is out-degree-zero in $\mathbf{B}$; also $\overline{F}a = Fa$ for each $a \in \mathbf{B}(B, *)$. Hence there is a unique $a \in \mathbf{B}(B, *)$ with $d = \overline{F}a$.

Suppose that B and $\operatorname{tgt} d$ are in $|\mathbf{B}| \setminus |\mathbf{A}|$. Then $\overline{F}B = B$, $d \in \mathbf{B}(B, *)$ and $\overline{F}d = d$. As $\overline{F}$ is injective on the morphisms of $\mathbf{B}$ not in $\mathbf{A}$, d is the unique morphism in $\mathbf{B}(B, *)$ mapped by $\overline{F}$ to d.

Otherwise, $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and $\operatorname{tgt} d \in |\mathbf{C}|$. Then $\overline{F}B = B$, and $d = [(c_1, b_1)]$ for some $A_1 \in |\mathbf{A}|$, some $b_1 \in \mathbf{B}(B, A_1)$ and some $c_1 \in \mathbf{C}(FA_1, C)$, where $C = \operatorname{tgt} d$. For uniqueness of lifts, suppose that $b_2 \in \mathbf{B}(B, *)$ is such that $d = \overline{F}b_2$. Let $A_2 = \operatorname{tgt} b_2$. Then $A_2 \in |\mathbf{A}|$ as $\overline{F}A_2 = \operatorname{tgt} d = C$, and so $\overline{F}b_2 = [(\mathrm{id}_C, b_2)]$. As $d = \overline{F}b_2$, we have $(\mathrm{id}_C, b_2) \sim (c_1, b_1)$. By Lemma 6.11, $b_2 = F^{A_2}\,\mathrm{id}_C \circ b_2 = F^{A_1}c_1 \circ b_1$; this determines b_2. For existence of lifts, note that $\overline{F}(F^{A_1}c_1 \circ b_1) = [(\mathrm{id}_C, F^{A_1}c_1 \circ b_1)] = [(FF^{A_1}c_1, b_1)] = [(c_1, b_1)] = d$. $\square$

Theorem 6.12. *The functor $\mathcal{G}$ creates pushouts of monic lenses with discrete opfibrations.*

Proof. Using the notation of Proposition 6.9, suppose that F is a discrete opfibration. Then $\overline{F}$ is also a discrete opfibration (Theorem 6.10). Let $J_{\mathbf{B}} \colon \mathbf{B} \to \mathbf{B} \sqcup \mathbf{C}$ and $J_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{B} \sqcup \mathbf{C}$ be the coproduct injection functors. Coproduct injections in $\mathcal{C}at$ are always discrete opfibrations, as is the coproduct copairing of any two discrete opfibrations. Hence $J_{\mathbf{B}}$, $J_{\mathbf{C}}$ and $[\bar{J}, \overline{F}]$ are all discrete opfibrations. As the composite of two discrete opfibrations is a discrete opfibration, so is $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$. So far, we know that $[\bar{J}, \overline{F}]$ is the coequaliser in $\mathcal{C}at$ of $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$, all of these functors have canonical lens structures as they are discrete opfibrations, and $[\bar{J}, \overline{F}]$ coforks $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$ in $\mathcal{L}ens$. As $[\bar{J}, \overline{F}]$ is a discrete opfibration, the conditions of Theorem 6.6 are satisfied, and so $[\bar{J}, \overline{F}]$ coequalises $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$ in $\mathcal{L}ens$. As $\mathcal{G}$ creates coproducts [8], it follows that $\bar{J}$ and $\overline{F}$ exhibit $\mathbf{D}$ as the pushout of J and F in $\mathcal{L}ens$. $\square$

One might hope that the above result generalises to pushouts of two discrete opfibrations, or of arbitrary lenses along monos; this is not the case. In the following example, we construct two discrete opfibrations whose pushout square in $\mathcal{C}at$ does not lie under a commuting square in $\mathcal{L}ens$.

Example 6.13. Let $\mathbf{A}$ and $\mathbf{B}$ be the categories generated respectively by the following graphs.

$$\begin{array}{c} Y_1' \xleftarrow{f_1'} X' \xrightarrow{f_2'} Y_2' \\ Y_1'' \xleftarrow[f_1'']{} X'' \xrightarrow[f_2'']{} Y_2'' \end{array} \qquad\qquad Y_1 \xleftarrow{f_1} X \xrightarrow{f_2} Y_2$$

Let $F\colon \mathbf{A} \to \mathbf{B}$ be the unique functor that sends both X' and X'' to X, both Y_1' and Y_1'' to Y_1, and both Y_2' and Y_2'' to Y_2. Let $G\colon \mathbf{A} \to \mathbf{B}$ be the unique functor that sends both X' and X'' to X, both Y_1' and Y_2'' to Y_1, and both Y_2' and Y_1'' to Y_2. Both F and G are discrete opfibrations. Their pushout in $\mathcal{C}at$ is $\mathbf{2}$; the pushout injections $\overline{F}, \overline{G}\colon \mathbf{B} \to \mathbf{2}$ are both the unique functor that sends X to 0, and both Y_1 and Y_2 to 1. There are two different lens structures on this functor; one lifts the unique morphism u of $\mathbf{2}$ to f_1, the other lifts it to f_2. This gives four different combinations of lens structures on $\overline{F}$ and $\overline{G}$. Assume, for a contradiction, that one of these combinations satisfies $\overline{F}G = \overline{G}F$ in $\mathcal{L}ens$. As $G^{X'}\overline{F}^X u = F^{X'}\overline{G}^X u$, we must have $\overline{F}^X u = \overline{G}^X u$. If $\overline{F}^X u = f_1$, then $G^{X''}\overline{F}^X u = G^{X''} f_1 = f_2'$ and $F^{X''}\overline{G}^X u = F^{X''} f_1 = f_1' \neq f_2'$, which is a contradiction. If $\overline{F}^X u = f_2$, we obtain a similar contradiction.

Next we give an example of a lens and a cosieve where the pushout of the get functor of the lens along the cosieve does not have a lens structure (incidentally this lens and cosieve do not have a pushout in $\mathcal{L}ens$).

Example 6.14. Let $\mathbf{B}$ and $\mathbf{D}$ be the preordered sets generated by the graphs

$$\begin{tikzcd} X \arrow[d, "s"'] & W \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d, "t"] \\ Z_2 & Z_1 \arrow[l] \arrow[r] & Z_3 \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[dr, "s'"'] & W' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d] & Y' \arrow[dl, "t'"] \\ & Z' & \end{tikzcd}$$

respectively. Let $\mathbf{A}$ be the out-degree-zero subcategory of $\mathbf{B}$ on the objects Z_1, Z_2 and Z_3, and let $J\colon \mathbf{A} \rightarrowtail \mathbf{B}$ be the inclusion lens. As $\mathbf{1}$ is terminal in $\mathcal{L}ens$ [8], there is a unique lens $F\colon \mathbf{A} \to \mathbf{1}$. By Proposition 6.9, the pushout of $\mathcal{G}F$ along $\mathcal{G}J$ in $\mathcal{C}at$ is the unique functor $\overline{F}\colon \mathbf{B} \to \mathbf{D}$ that maps W to W', X to X', Y to Y', and all of Z_1, Z_2 and Z_3 to Z'. The functor $\overline{F}$ has no lens structure, otherwise we could derive the contradiction

$$s \circ f = \overline{F}^X s' \circ \overline{F}^W f' = \overline{F}^W (s' \circ f') = \overline{F}^W (t' \circ g') = \overline{F}^Y t' \circ \overline{F}^W g' = t \circ g.$$

From Theorem 6.12, every monic lens has a cokernel pair. Actually, using the epi-mono factorisation, every lens has a cokernel pair, namely, the cokernel pair of its mono factor.

Proposition 6.15. *Every monic lens is effective (i.e. equalises its cokernel pair).*

Proof. Let $M\colon \mathbf{A} \to \mathbf{B}$ be a monic lens, and let $J_1, J_2\colon \mathbf{B} \to \mathbf{Coker}\, M$ be its cokernel pair. Based on Proposition 6.9, if $B \in |\mathbf{B}|$ is such that $J_1 B = J_2 B$, then $B \in |\mathbf{A}|$; and similarly for morphisms of $\mathbf{B}$. In particular, the image of any lens which forks J_1 and J_2 is contained in $\mathbf{A}$, and thus its corestriction to $\mathbf{A}$ is the unique comparison lens. □

Corollary 6.16. *In $\mathcal{L}ens$, the classes of all monos, effective monos, regular monos, strong monos and extremal monos coincide.*

Corollary 6.17. *Every lens that is both epic and monic is an isomorphism.*

6.4. Regular epic lenses

In this section, we show that all epis in $\mathcal{L}ens$ are regular. This implies that the epic lenses form another class of coequalisers in $\mathcal{L}ens$. For contrast, recall that not all epis in $\mathcal{C}at$ are regular.

Example 6.18. In Example 5.11, we saw that the functor $J\colon \mathbf{2} \to \mathbf{I}$ is epic. It is, however, not a regular epi. Indeed, if J coforks $F_1, F_2\colon \mathbf{A} \to \mathbf{2}$, then $F_1 = F_2$ as J is monic, and so $\mathrm{id}_{\mathbf{2}}$ is the coequaliser of F_1 and F_2, but $\mathbf{2}$ and $\mathbf{I}$ are not isomorphic.

Proposition 6.19. *The get functor of every epic lens is an effective epi in $\mathcal{C}at$.*

A functor $E\colon \mathbf{B} \to \mathbf{C}$ is *surjective on composable pairs* if for each composable pair (c, c') of $\mathbf{C}$, there is a composable pair (b, b') of $\mathbf{B}$ such that $Eb = c$ and $Eb' = c'$; such functors are necessarily also surjective on objects and morphisms. If $E\colon \mathbf{B} \to \mathbf{C}$ is an epic lens, then $\mathcal{G}E$ is surjective on composable pairs; indeed, if (c, c') is a composable pair of $\mathbf{C}$, then there is a $B \in |\mathbf{B}|$ such that $EB = \operatorname{src} c$, and $(E^B c, E^{\operatorname{tgt} E^B c} c')$ is a composable pair above (c, c'). Hence it suffices to prove the following lemma.

Lemma 6.20. *All functors that are surjective on composable pairs are effective epis in $\mathcal{C}at$.*

Proof. Let $E\colon \mathbf{B} \to \mathbf{C}$ be a surjective-on-composable-pairs functor. Let $F_1, F_2\colon \mathbf{K} \to \mathbf{B}$ be a kernel pair of E. We must show that E coequalises F_1 and F_2. Let $G\colon \mathbf{B} \to \mathbf{D}$ cofork F_1 and F_2.

Suppose that there is a functor $H\colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$. As E is surjective on objects, for all $C \in |\mathbf{C}|$ there is a $B \in |\mathbf{B}|$ such that $EB = C$, and so $HC = HEB = GB$; this equation determines H on objects. As E is surjective on morphisms, a similar equation determines H on morphisms.

To define $H\colon \mathbf{C} \to \mathbf{D}$ with these equations, the values of GB and Gb should be independent of the choice of B above C and b above c. For all $C \in |\mathbf{C}|$ and all $B, B' \in |\mathbf{B}|$ such that $EB = EB' = C$, we have $GB = GF_1\langle B,\, B'\rangle = GF_2\langle B,\, B'\rangle = GB'$, where $\langle B,\, B'\rangle \in |\mathbf{K}|$ comes from the pullback property; hence the object map of H is well defined. Its morphism map is similarly also well defined.

Define H with the above equations. We must show that H is a functor. For all $C \in |\mathbf{C}|$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $H\,\mathrm{id}_C = G\,\mathrm{id}_B = \mathrm{id}_{GB} = \mathrm{id}_{HC}$; thus H preserves identities. For all composable pairs c and c' of $\mathbf{C}$, there is a composable pair b and b' of $\mathbf{B}$ such that $Eb = c$ and $Eb' = c'$, and $H(c' \circ c) = G(b' \circ b) = Gb' \circ Gb = Hc' \circ Hc$; thus H preserves composites. By construction, $G = H \circ E$. □

Corollary 6.21. *Every epic lens coequalises its proxy kernel pair, and so is regular.*

Proof. Let $E\colon \mathbf{B} \to \mathbf{C}$ be an epic lens. Let $F_1, F_2\colon \mathbf{K} \to \mathbf{B}$ be a proxy kernel pair of E in $\mathcal{L}ens$. By Proposition 6.19, $\mathcal{G}E$ coequalises $\mathcal{G}F_1$ and $\mathcal{G}F_2$ in $\mathcal{C}at$. Let $G\colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2, let $B \in |\mathbf{B}|$, let $d \in \mathbf{D}(GB, *)$, and let $C = EB$. Then

$$(G \circ F_1)^{\langle B,\, B\rangle} d = {F_1}^{\langle B,\, B\rangle} G^B d = \big\langle G^B d,\, E^B E G^B d\big\rangle,$$

and similarly $(G \circ F_2)^{\langle B,\, B\rangle} d = \big\langle E^B E G^B d,\, G^B d\big\rangle$. As G coforks F_1 and F_2, it follows that $G^B d = E^B E G^B d$. By Theorem 6.6, E coequalises F_1 and F_2 in $\mathcal{L}ens$. □

Corollary 6.22. *In $\mathcal{L}ens$, the classes of all epis, proxy effective epis, regular epis, strong epis and extremal epis coincide.*

In Remark 5.8, we promised a proof of the orthogonality conditions for the (epi, mono) factorisation system on $\mathcal{L}ens$. By Corollary 6.22, every epic lens is a strong epi. Conversely, as $\mathcal{L}ens$ has equalisers [8], each lens that is left orthogonal to all monic lenses is an epic lens. Hence the epic lenses form the class of morphisms in $\mathcal{L}ens$ that are left orthogonal to all monic lenses. Similarly, every monic lens is a strong mono by Corollary 6.16. To deduce the other orthogonality condition, it remains to prove the following proposition.

Proposition 6.23. *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a lens such that every epic lens is left orthogonal to F. Then F is monic.*

Proof. Let $G_1, G_2\colon \mathbf{C} \to \mathbf{A}$ be lenses such that $F \circ G_1 = F \circ G_2$. Let $F = M \circ E$ be the factorisation of F from Proposition 5.7. As $E \perp F$, there is a unique lens $H\colon \mathbf{Im}\, F \to \mathbf{A}$ such that $H \circ E = \mathrm{id}_{\mathbf{A}}$ and $M = F \circ H$. As $M \circ E \circ G_1 = F \circ G_1 = F \circ G_2 = M \circ E \circ G_2$ and M is monic, actually $E \circ G_1 = E \circ G_2$. Thus $G_1 = H \circ E \circ G_1 = H \circ E \circ G_2 = G_2$. $\square$

CHAPTER 7

Conclusion

As noted in the introduction, a serious study of the categorical properties of the category $\mathcal{L}ens$ of asymmetric (delta) lenses had not been attempted until the work of Chollet et al. [8] and the work in this thesis. The proxy pullback in $\mathcal{L}ens$, a notion introduced by Johnson and Rosebrugh to describe composition of symmetric (delta) lenses when viewed as spans of asymmetric ones [17], ended up playing an important role in the proofs of several of these categorical properties. These proofs were often inspired by how the real pullback is used to prove these properties of other categories. Naturally, questions about the other ways in which the proxy pullback might behave like a real pullback arose, and these led to the rest of the work in this thesis.

In the remainder of this chapter, we recall the important contributions of this thesis, and suggest ideas for how this work may be continued. Substantial progress has already been made for many of these ideas.

7.1. Proxy pullbacks

In Chapter 3, we gave a new perspective on proxy pullbacks in $\mathcal{L}ens$—a proxy pullback is a compatible square of lenses whose get functors form a pullback square in $\mathcal{C}at$. We also proved several properties of proxy pullbacks that mirror those of real pullbacks, including

- that compatible-lens transformations between lens cospans extend uniquely to compatible-lens transformations between chosen proxy pullbacks of the cospans,
- that proxy pullbacks are unique up to unique span isomorphism,
- a proxy analogue of the pullback pasting lemma, and
- that proxy kernel pairs are equipped with canonical symmetry and transitivity lenses.

All of the major results in Chapter 3, including those listed above, were proved in terms of diagrams of compatible squares. This approach ultimately relied on

- the ability to paste compatible squares of functors and cofunctors horizontally and vertically,
- the fact that lenses and discrete opfibrations may be equivalently defined in terms of compatible squares of functors and cofunctors of a certain shape, and
- Theorem 3.6, which said that the functor $\langle \mathcal{S}, \mathcal{T} \rangle \colon \mathcal{C}of\mathcal{S}q \to \mathcal{C}at \times \mathcal{C}at$ creates limits.

A reader familiar with the notion of a double category has likely already deduced that categories, functors, cofunctors and compatible squares are the objects, arrows, proarrows and cells of a flat strict double category $\mathbb{C}of$; the two ways to paste compatible squares of functors and cofunctors together give the two kinds of cell composition. Inspired by this observation, we may define the category $\mathcal{L}ens\,\mathbb{C}$ of lenses in a flat strict double category $\mathbb{C}$ to be the category whose objects are the objects of $\mathbb{C}$ and whose morphisms $(p, g)\colon X \to Y$ are the cells in $\mathbb{C}$ of shape

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"', "\mid" marking] \arrow[dr, phantom, "\Rightarrow"] & Y \arrow[d, equal, "\mid" marking] \\ Y \arrow[r, equal] & Y \end{tikzcd} .$$

One may then define the notion of compatible square of lenses in $\mathbb{C}$ using the cells in $\mathbb{C}$ for the compatibility conditions, and from this, also define the notion of proxy pullback in terms of pullbacks in the category $\mathbb{C}_0$ of objects and arrows in $\mathbb{C}$. If we also ask that the functor $\langle \mathcal{S}, \mathcal{T} \rangle\colon \mathbb{C}_1 \to \mathbb{C}_0 \times \mathbb{C}_0$ creates limits, where $\mathcal{S}$ and $\mathcal{T}$ are the source and target functors from the category $\mathbb{C}_1$ of proarrows and cells of $\mathbb{C}$ to the category $\mathbb{C}_0$ of objects and arrows of $\mathbb{C}$, then all of the major results about proxy pullbacks in the remainder of Chapter 3 should generalise to this setting with essentially the same proofs.

It is also interesting to note that the *companion pairs* in $\mathbb{C}of$ are the discrete opfibrations whilst the *conjoint pairs* in $\mathbb{C}of$ are the bijective-on-objects functors. Clarke [9] has observed that every cofunctor is given, essentially uniquely, by a span of functors whose left leg is a bijective-on-objects functor and whose right leg is a discrete opfibration. Viewed from our double category perspective, $\mathbb{C}of$ has what are called *effective tabulators* by Paré [26] and *strong tabulators* by Lambert [23]; such tabulators are defined in terms of the companions and conjoints of the ambient double category. Thus if our double category $\mathbb{C}$ has effective tabulators then the generalised cofunctors have such a representation as a span whose left leg is a generalised bijective-on-objects functor and whose right leg is a generalised discrete opfibration.

Many approaches to generalised category theory fit into the framework of monads and monad morphisms in a (pseudo) double category; monads give the appropriate notion of generalised category and monad morphisms give the appropriate notion of generalised functor. For category theory internal to a category $\mathbf{C}$ with pullbacks, consider the double category $\mathbb{S}pan\,\mathbf{C}$ of objects, morphisms and spans in $\mathbf{C}$. For category theory enriched in a nice-enough monoidal category $\mathcal{V}$, consider the double category $\mathbb{M}at\,\mathcal{V}$ of sets, functions and $\mathcal{V}$-valued matrices. Finally, for category theory internal to a sufficiently nice monoidal category $\mathcal{V}$, consider the double category $\mathbb{C}omod\,\mathcal{V}$ of comonoids, comonoid morphisms and bicomodules in $\mathcal{V}$ [1]. Less well known is that for a double category $\mathbb{D}$ with companions, there is a second kind of morphism between monads, for which the author proposes the name *monad retromorphism*, that is best defined as a *retrocell* [27] between the underlying proarrows of the monads that preserves the unit and multiplication cells of the monads in the appropriate way. These monad retromorphisms give the appropriate notion of generalised cofunctor. For this reason, and due to the common confusion between the notions of *cofunctor* and *contravariant functor*, the author also proposes that we rename *cofunctor* to *retrofunctor*, although the author is also aware that it is perhaps too late for this new name to be widely adopted. Actually, for any double category $\mathbb{D}$ with companions, there is a flat strict double category $\mathbb{M}nd\,\mathbb{D}$ of monads, monad morphisms and monad retromorphisms[1]; the cells in this double category give the appropriate generalised notion of compatible square.

[1]The double category $\mathbb{M}nd\,\mathbb{D}$ is a sub-double-category of the one defined by Fiore, Gambino, and Kock [13].

The generalised lenses in $\mathbb{M}nd\,\mathbb{D}$ for a double category $\mathbb{D}$ with companions more closely resemble the primitive notion of lens than the generalised lenses in some arbitrary flat strict double category $\mathbb{C}$. Indeed, letting $\mathbb{D}$ be $\mathbb{S}pan\,\mathbf{C}$, $\mathbb{M}at\,\mathcal{V}$ or $\mathbb{C}omod\,\mathcal{V}$, we obtain the right notion of lens in each of the corresponding generalised category theories. For a more exotic example, consider the flat strict double category $\mathbb{F}ltr$ of sets, functions and Kleisli arrows of the filter monad $\mathcal{F}$ on $\mathcal{S}et$, where a cell

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{N}"', "|" marking] \arrow[dr, phantom, "\Rightarrow"] & Y \arrow[d, "\mathcal{M}", "|" marking] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

exists if and only if $(\mathcal{F}f')\mathcal{N}_x \supseteq \mathcal{M}_{fx}$ for all $x \in X$. The double category $\mathbb{M}nd\,\mathbb{F}ltr$ is then the flat strict double category $\mathbb{T}op$ of topological spaces, continuous maps and open maps, where a cell exists exactly when the square of functions underlying its boundary commutes. Notice that preimages for continuous maps and direct images for open maps are like the action of functors and cofunctors on morphisms; this is not surprising because the full subcategory of $\mathcal{C}at$ spanned by the preorders is isomorphic to that of the category $\mathcal{T}op$ of topological spaces and continuous maps spanned by the Alexandrov-discrete spaces. A generalised lens in $\mathbb{T}op$ is then an open continuous map. The proxy pullback of a cospan of open continuous maps is given by the pullback in $\mathcal{T}op$ of the cospan regarded as merely a cospan of continuous maps; the pullback projections turn out to also be open maps. It appears as though $\mathbb{T}op$ does not have effective tabulators. Understanding the conditions on a double category $\mathbb{D}$ with companions such that the source and target functors of $\mathbb{M}nd\,\mathbb{D}$ satisfy the limit creation property described earlier, and also the conditions on $\mathbb{D}$ such that $\mathbb{M}nd\,\mathbb{D}$ has effective tabulators, is ongoing work.

Taking seriously the idea that compatibility is a generalised notion of commutativity for mixed diagrams of a certain shape, a natural question to ask is whether there is a useful notion of compatibility for arbitrarily shaped mixed diagrams. This is another direction for future work.

7.2. Universal properties of the proxy pullback

In Chapter 4, we established necessary and sufficient conditions for when a lens span that forms a commuting square with a lens cospan has a comparison lens to a proxy pullback of the cospan. These conditions involved the new notions of sync minimality and independence, as well as the notion of compatibility from Chapter 3. They enabled us to describe exactly when a proxy pullback is a real pullback, and this description simplified even further when merely considering proxy products. A search for such a simplified description for general proxy pullbacks is ongoing.

In order to lift the theory of Chapter 4 to the setting of generalised lenses as described in the previous section, a first step would be to obtain categorical characterisations of the notions of sync minimality and independence, perhaps in terms of some universal property. Whilst the author is yet to discover a compelling such characterisation of independent lens spans, some interesting progress has already been made for sync minimal ones. The key observation is that being sync minimal is really a property of the put cofunctors of a lens span, and that the process of taking the sync minimal core actually gives a factorisation of this span of put cofunctors.

Spivak and Niu [30] show that $\mathcal{C}of$ has products; the explicit description of these products is unfortunately rather complicated—the objects of the product of two categories are certain pairs of rooted infinite trees whose edges are morphisms from either category, and the morphisms out of such an object are the paths in either tree from its root. It turns out that a cofunctor span is sync minimal exactly when its product pairing in $\mathcal{C}of$ has surjective lifting functions. We will call a cofunctor with surjective lifting functions *cofull* and one with injective lifting functions *cofaithful*. There is a well-known factorisation system on $\mathcal{C}of$ whose left class is the bijective-on-objects cofunctors and whose right class is the discrete opfibrations [10], which, in this context, we might also call the cofully cofaithful cofunctors. The factorisation system on $\mathcal{C}of$ that we are actually interested in has as its left class the cofaithful bijective-on-objects cofunctors, and its right class the cofull cofunctors; this factorisation of the put cofunctor of a lens coincides with the other factorisation. If we factor the product pairing of a cofunctor span using this factorisation system, then the sync-minimal core of the cospan is obtained by composing the second factor with the appropriate product projection cofunctors.

We have already recalled that symmetric lenses between two categories correspond to the equivalence classes of a certain equivalence relation on asymmetric lens spans between the two categories [17]. Clarke, with a different definition of symmetric lens, constructed an adjoint triple[2]

$$\begin{tikzcd} \mathcal{S}ym\mathcal{L}ens(\mathbf{A},\mathbf{B}) \arrow[r, shift left=6, "\mathcal{L}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\mathcal{R}"'] & \mathcal{S}pan\mathcal{L}ens(\mathbf{A},\mathbf{B}) \arrow[l, "\mathcal{M}" description] \end{tikzcd}$$

between his category $\mathcal{S}ym\mathcal{L}ens(\mathbf{A},\mathbf{B})$ of symmetric lenses from $\mathbf{A}$ to $\mathbf{B}$ and the category $\mathcal{S}pan\mathcal{L}ens(\mathbf{A},\mathbf{B})$ whose objects are lens spans from $\mathbf{A}$ to $\mathbf{B}$ and whose morphisms are functors satisfying certain compatibility conditions [9]. The comonad $\mathcal{L} \circ \mathcal{M}$ on $\mathcal{S}pan\mathcal{L}ens(\mathbf{A},\mathbf{B})$ induced by the adjoint triple appears to be closely related to our process that sends a lens span to its sync minimal core. Additionally, as $\mathcal{L}$ is fully faithful, we may think of those lens spans in the image of $\mathcal{L}$ as representing symmetric lenses. It might thus be reasonable to think of the sync-minimal lens spans as being the symmetric lenses, an idea that is reinforced by the interpretation of the sync-minimal property that was given in Section 4.1.

The original proposal for the *Categories of Maintainable Relations* project of the Applied Category Theory Adjoint School 2020, which did not end up being the actual focus of the project, was to work out how to view symmetric lenses as some kind of generalised relations in $\mathcal{L}ens$. A *relation* in a category from object X to object Y is usually defined as a jointly monic span from X to Y. A *regular category* [4] is a finitely complete category with a pullback-stable regular-epi mono factorisation system. Relations in regular categories are particularly nice as they form the morphisms of a bicategory; the composite of two relations is the image (from the factorisation system) of their composite as spans (from the pullback). Given a not-necessarily-proper orthogonal factorisation system $(\mathscr{E}, \mathscr{M})$ on a category with products, an $\mathscr{M}$-relation from X to Y is a span from X to Y whose product pairing is in $\mathscr{M}$. If the factorisation system is pullback-stable, then the $\mathscr{M}$-relations still form the morphisms of a bicategory with nice

[2] Clarke's functor $\mathcal{M}$ is not to be confused with our $\mathcal{M}$ that sends a lens span to its sync-minimal core.

properties [22, 24, 28], where composition of $\mathscr{M}$-relations is defined similarly to that of relations in a regular category. As $\mathcal{C}of$ is finitely complete, we may consider the $\mathscr{M}$-relations in $\mathcal{C}of$ for the factorisation system where $\mathscr{E}$ is the class of cofaithful bijective-on-objects cofunctors and $\mathscr{M}$ is the class of cofull cofunctors. From our earlier discussion, these $\mathscr{M}$-relations are exactly the sync-minimal cofunctor spans. It would be interesting to work out what the composition of such $\mathscr{M}$-relations is, as the pullback in $\mathcal{C}of$ is very different to the proxy pullback in $\mathcal{L}ens$. Returning to the original question of whether symmetric lenses may be viewed as some kind of relations in $\mathcal{L}ens$, we seem to need a further generalisation of the notion of internal relation as the sync-minimal core of a lens span is not obtained from a factorisation system on $\mathcal{L}ens$ itself.

7.3. Categorical properties of the category of asymmetric lenses

In Chapters 5 and 6, the contents of which will appear in the Applied Category Theory 2021 conference proceedings [11], we studied several categorical properties of $\mathcal{L}ens$ including its monos and epis, and its coequalisers. We gave a complete elementary characterisation of the monos and epis in $\mathcal{L}ens$, the monos being the unique lenses on cosieves and the epis being the surjective on objects lenses. From this, we saw that Johnson and Roseburgh's factorisation system on $\mathcal{L}ens$ [18] is actually an epi-mono factorisation system. We also initiated a study of the coequalisers in $\mathcal{L}ens$. Despite $\mathcal{L}ens$ not having all coequalisers, nor the forgetful functor from $\mathcal{L}ens$ to $\mathcal{C}at$ preserving or reflecting them, we presented two interesting positive results. First, every epic lens coequalises its proxy kernel pair. Second, $\mathcal{L}ens$ has pushouts of discrete opfibrations along cosieves. Our characterisation of the epic lenses played a central role in the proof of both of these results, and hopefully will enable future work to completely characterise the coequalisers in $\mathcal{L}ens$.

Finding useful axiomatisations of pullback-like constructions is a nascent area of research. Noteworthy attempts, all of which we have already mentioned, include Simpson's *local independent products* [29], Böhm's *relative pullbacks* [6], and Bumpus and Kocsis' *proxy pushout* [7]. A good axiomatisation is one that is general enough to include many of the known examples, but specific enough to have a rich theory. Neither the notion of local independent product nor the notion of relative pullback is general enough to include proxy pullbacks in $\mathcal{L}ens$, although it is conceivable that a small modification of either notion could include the sync-minimal proxy pullbacks. It also appears as though the axiomatisation of proxy pushouts given by Bumpus and Kocsis is too general to have a rich theory. One possible goal for an axiomatisation of proxy pullbacks would be to have a theory of *proxy-regular categories* that mirrors the theory of regular categories. The category $\mathcal{L}ens$ is an obvious example of such a proxy-regular category from which we may draw inspiration; the category of topological spaces and open continuous maps is another likely candidate.

Bibliography

[1] M. Aguiar. "Internal Categories and Quantum Groups". PhD thesis. Cornell University, 1997. URL: http://pi.math.cornell.edu/~maguiar/thesis2.pdf.

[2] D. Ahman and T. Uustalu. "Taking Updates Seriously". In: *Proceedings of the 6th International Workshop on Bidirectional Transformations* (Uppsala, Sweden). Ed. by R. Eramo and M. Johnson. 2017, pp. 59–73. URL: http://ceur-ws.org/Vol-1827/.

[3] F. Bancilhon and N. Spyratos. "Update Semantics of Relational Views". In: *ACM Transactions on Database Systems* 6.4 (1981), pp. 557–575. DOI: 10.1145/319628.319634.

[4] M. Barr, P. A. Grillet, and D. H. van Osdol. *Exact Categories and Categories of Sheaves*. 1st ed. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1971. DOI: 10.1007/BFb0058579.

[5] M. A. Bednarczyk, A. M. Borzyszkowski, and W. Pawlowski. "Generalized congruences — Epimorphisms in Cat". In: *Theory and Applications of Categories* 5.11 (1999), pp. 266–280. URL: http://www.tac.mta.ca/tac/volumes/1999/n11/5-11abs.html.

[6] G. Böhm. "Crossed Modules of Monoids I: Relative Categories". In: *Applied Categorical Structures* 27.6 (2019), pp. 641–662. DOI: 10.1007/s10485-019-09570-0.

[7] B. M. Bumpus and Z. A. Kocsis. *Spined categories: generalizing tree-width beyond graphs*. 2021. arXiv: 2104.01841.

[8] E. Chollet, B. Clarke, M. Johnson, M. Songa, V. Wang, and G. Zardini. *Limits and colimits in a category of lenses*. To appear in: Applied Category Theory 2021. URL: https://arxiv.org/abs/2105.05422.

[9] B. Clarke. "A diagrammatic approach to symmetric lenses". In: Proceedings of the 3rd Annual International *Applied Category Theory Conference 2020,* Cambridge, USA, 6–10th July 2020. Ed. by D. I. Spivak and J. Vicary. Vol. 333. Electronic Proceedings in Theoretical Computer Science. Open Publishing Association, 2021, pp. 79–91. DOI: 10.4204/EPTCS.333.6.

[10] B. Clarke. "Internal lenses as functors and cofunctors". In: Proceedings *Applied Category Theory 2019,* University of Oxford, UK, 15–19 July 2019. Ed. by J. Baez and B. Coecke. Vol. 323. Electronic Proceedings in Theoretical Computer Science. 2020, pp. 183–195. DOI: 10.4204/EPTCS.323.13.

[11] M. Di Meglio. *Coequalisers under the lens*. To appear in: Applied Category Theory 2021.

[12] Z. Diskin, Y. Xiong, and K. Czarnecki. "From State- to Delta-Based Bidirectional Model Transformations: the Asymmetric Case". In: *Journal of Object Technology* 10 (2011), 6:1–25. DOI: 10.5381/jot.2011.10.1.a6.

[13] T. M. Fiore, N. Gambino, and J. Kock. "Monads in double categories". In: *Journal of Pure and Applied Algebra* 215.6 (2011), pp. 1174–1197. DOI: https://doi.org/10.1016/j.jpaa.2010.08.003.

[14] J. N. Foster, M. B. Greenwald, J. T. Moore, B. C. Pierce, and A. Schmitt. "Combinators for Bidirectional Tree Transformations: A Linguistic Approach to the View-Update

Problem". In: *ACM Transactions on Programming Languages and Systems* 29.3 (2007). DOI: 10.1145/1232420.1232424.

[15] R. Fritsch and D. M. Latch. "Homotopy Inverses for Nerve". In: *Mathematische Zeitschrift* 177.2 (1981), pp. 147–179. DOI: 10.1007/BF01214196.

[16] J. Gibbons and P. Stevens, eds. *Bidirectional Transformations*. Vol. 9715. Lecture Notes in Computer Science. Springer International Publishing, 2018. DOI: 10.1007/978-3-319-79108-1.

[17] M. Johnson and R. Rosebrugh. "Spans of Delta Lenses". In: *Proceedings of the 4th International Workshop on Bidirectional Transformations* (L'Aquila, Italy). Ed. by A. Cunha and E. Kindler. 2015, pp. 1–15. URL: http://ceur-ws.org/Vol-1396/.

[18] M. Johnson and R. Rosebrugh. "The more legs the merrier: A new composition for symmetric (multi-)lenses". In: Proceedings of the 3rd Annual International *Applied Category Theory Conference 2020,* Cambridge, USA, 6–10th July 2020. Ed. by D. I. Spivak and J. Vicary. Vol. 333. Electronic Proceedings in Theoretical Computer Science. Open Publishing Association, 2021, pp. 92–107. DOI: 10.4204/EPTCS.333.7.

[19] M. Johnson and R. Rosebrugh. "Unifying Set-Based, Delta-Based and Edit-Based Lenses". In: *Proceedings of the 5th International Workshop on Bidirectional Transformations* (Eindhoven, The Netherlands). Ed. by A. Anjorin and J. Gibbons. 2016, pp. 1–13. URL: http://ceur-ws.org/Vol-1571/.

[20] M. Johnson and R. Rosebrugh. "Universal Updates for Symmetric Lenses". In: *Proceedings of the 6th International Workshop on Bidirectional Transformations* (Uppsala, Sweden). Ed. by R. Eramo and M. Johnson. 2017, pp. 39–53. URL: http://ceur-ws.org/Vol-1827/.

[21] P. T. Johnstone. *Sketches of an Elephant — A Topos Theory Compendium*. Vol. 1. Oxford University Press, 2002. ISBN: 9780198534259.

[22] G. M. Kelly. "A note on relations relative to a factorization system". In: *Category Theory*. Ed. by A. Carboni, M. C. Pedicchio, and G. Rosolini. Berlin, Heidelberg: Springer Berlin Heidelberg, 1991, pp. 249–261.

[23] M. Lambert. *Characterizing Double Categories of Relations*. To appear in: Applied Category Theory 2021.

[24] J. Meisen. "On Bicategories of Relations and Pullback Spans". In: *Communications in Algebra* 1.5 (1974), pp. 377–401. DOI: 10.1080/00927877408548625.

[25] F. J. Oles. "A Category-Theoretic Approach to the Semantics of Programming Languages". PhD thesis. Syracuse University, 1982. URL: https://www.cs.cmu.edu/afs/cs.cmu.edu/project/fox-19/member/jcr/www/FrankOlesThesis.pdf.

[26] R. Paré. *A double category take on restriction categories*. CT2018. Azores, Portugal. URL: http://www.mat.uc.pt/~ct2018/slides/R_Pare.pdf.

[27] R. Paré. *Retrocells*. CT2019. Edinburgh, Scotland. URL: https://conferences.inf.ed.ac.uk/ct2019/slides/28.pdf.

[28] D. Pavlović. "Maps I: relative to a factorisation system". In: *Journal of Pure and Applied Algebra* 99.1 (1995), pp. 9–34. DOI: https://doi.org/10.1016/0022-4049(94)00054-M.

[29] A. Simpson. "Category-theoretic Structure for Independence and Conditional Independence". In: *Electronic Notes in Theoretical Computer Science* 336 (2018), pp. 281–297. DOI: 10.1016/j.entcs.2018.03.028.

[30] D. I. Spivak and N. Niu. *Polynomial Functors: A General Theory of Interaction*. 2021. URL: https://topos.site/poly-book.pdf.